Global Smooth Effects, Well-Posedness and Scattering for the Derivative Nonlinear Schrödinger Equation with Small Rough Data

Wang Baoxiang, Han Lijia, Huang Chunyan

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China

E-mails: wbx, hanlijia, hcy@math.pku.edu.cn

Abstract

We obtain the global smooth effects for the solutions of the linear Schrödinger equation in anisotropic Lebesgue spaces. Applying these estimates, we study the Cauchy problem for the generalized elliptical and non-elliptical derivative nonlinear Schrödinger equations (DNLS) and get the global well posedness of solutions with small data in modulation spaces $M_{2,1}^{3/2}(\mathbb{R}^n)$. Noticing that $B_{2,1}^{s+n/2} \subset M_{2,1}^s \subset B_{2,1}^s$ are optimal inclusions, we have shown the global well posedness of DNLS with a class of rough data. As by products, the existence of the scattering operators with small data is also obtained.

Keywords. Derivative nonlinear Schrödinger equation, global smooth effects, global well posedness, small data.

MSC: 35 Q 55, 46 E 35, 47 D 08.

1 Introduction

This paper is a continuation of our earlier work [32] and we study the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gDNLS)

$$iu_t + \Delta_{\pm}u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x), \tag{1.1}$$

where u is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta_{\pm}u = \sum_{i=1}^{n} \varepsilon_i \partial_{x_i}^2, \quad \varepsilon_i \in \{1, -1\}, \quad i = 1, ..., n, \tag{1.2}$$

$\nabla = (\partial_{x_1}, ..., \partial_{x_n})$, $F : \mathbb{C}^{2n+2} \to \mathbb{C}$ is a polynomial series,

$$F(z) = F(z_1, ..., z_{2n+2}) = \sum_{m+1<|\beta|<\infty} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}, \tag{1.3}$$

$2 \le m < \infty$, $m \in \mathbb{N}$, $\sup_\beta |c_\beta| < \infty$[1]. A typical nonlinear term is the following

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = |u|^2 \vec{\lambda} \cdot \nabla u + u^2 \vec{\mu} \cdot \nabla \bar{u} + |u|^2 u,$$

which is a model equation in the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics [26, 10, 8]. Another typical nonlinearity is

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = (1 \mp |u|^2)^{-1} |\nabla u|^2 u = \sum_{k=0}^{\infty} \pm |u|^{2k} |\nabla u|^2 u, \quad |u| < 1,$$

which is a deformation of the Schrödinger map equation [9, 12].

A large amount of work has been devoted to the study of the local and global well posedness of (1.1), see Bejenaru and Tataru [2], Chihara [3, 4], Kenig, Ponce and Vega [14, 15], Klainerman [18], Klainerman and Ponce [19], Ozawa and Zhai [21], Shatah [22], B. Wang and Y. Wang [32]. When the nonlinear term F satisfies some energy structure conditions, or the initial data suitably decay, the energy method, which went back to the work of Klainerman [18] and was developed in [3, 4, 19, 21, 22], yields the global existence of (1.1) in the elliptical case $\Delta_\pm = \Delta$. Recently, Ozawa and Zhai obtained the global well posedness in $H^s(\mathbb{R}^n)$ ($n \geqslant 3$, $s > 2 + n/2$, $m \geqslant 2$) with small data for (1.1) in the elliptical case, where an energy structure condition on F is still required.

By setting up the local smooth effects for the solutions of the linear Schrödinger equation, Kenig, Ponce and Vega [14, 15] were able to deal with the non-elliptical case and they established the local well posedness of Eq. (1.1) in H^s with $s \gg n/2$. Recently, the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [16, 17].

In one spatial dimension, B. Wang and Y. Wang [32] showed the global well posedness of gDNLS (1.1) for small data in critical Besov spaces $\dot{B}_{2,1}^{1+n/2-2/m} \cap \dot{B}_{2,1}^{1+n/2-1/M}(\mathbb{R})$, $m \geqslant 4$. In higher spatial dimensions $n \geqslant 2$, by using Kenig, Ponce and Vega's local smooth effects and establishing time-global maximal function estimates in space-local Lebesgue spaces, B. Wang and Y. Wang [32] showed the

---

[1]In fact, c_β is not necessarily bounded, condition $\sup_\beta |c_\beta| < \infty$ can be replaced by $|c_\beta| \le C^{|\beta|}$.

global well posedness of gDNLS (1.1) for small data in Besov spaces $B_{2,1}^s(\mathbb{R}^n)$ with $s > n/2 + 3/2$, $m \geqslant 2 + 4/n$.

Wang and Huang [31] obtained the global well posedness of (1.1) in one spatial dimension with initial data in $M_{2,1}^{1+1/m}$, $m \geqslant 4$. In this paper, we will use a new way to study the global well posedness of (1.1) and show that (1.1) is globally well posed in $M_{2,1}^s(\mathbb{R}^n)$ with $s \geqslant 3/2$, $m \geqslant 2$ and $m > 4/n$ for the small Cauchy data. Our starting point is the smooth effect estimates for the linear Schrödinger equation in one spatial dimension (cf. [7, 13, 14, 23, 33]), from which we get a series of linear estimates in higher dimensional anisotropic Lebesgue spaces, including the global smooth effect estimates, the maximal function estimates and their relations to the Strichartz estimates. The maximal function estimates follows an idea as in Ionescu and Kenig [12]. These estimates together with the frequency-uniform decomposition method yield the global well posedness and scattering of solutions in modulation spaces $M_{2,1}^s$, $s \geqslant 3/2$.

1.1 $M_{2,1}^s$ and $B_{2,1}^s$

In this paper, we are mainly interested in the cases that the initial data u_0 belongs to the modulation space $M_{2,1}^s$ for which the norm can be equivalently defined in the following way (cf. [11, 29, 30, 31]):

$$\|f\|_{M_{2,1}^s} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\mathscr{F} f\|_{L^2(Q_k)}, \tag{1.4}$$

where $\langle k \rangle = 1 + |k|$, $Q_k = \{\xi : -1/2 \leq \xi_i - k_i < 1/2, \ i = 1, ..., n\}$. For simplicity, we write $M_{2,1} = M_{2,1}^0$. Since only the modulation space $M_{2,1}^s$ will be used in this paper, we will not state the defination of the general modulation spaces $M_{p,q}^s$, one can refer to Feichtinger [11]. Modulation spaces $M_{2,1}^s$ are related to the Besov spaces $B_{2,1}^s$ for which the norm is defined as follows:

$$\|f\|_{B_{2,1}^s} = \|\mathscr{F} f\|_{L^2(B(0,1))} + \sum_{j=1}^{\infty} 2^{sj} \|\mathscr{F} f\|_{L^2(B(0,2^j) \backslash B(0,2^{j-1}))}, \tag{1.5}$$

where $B(x_0, R) := \{\xi \in \mathbb{R} : |\xi - x_0| \leq R\}$. It is known that there holds the following optimal inclusions between $B_{2,1}^{n/2+s}$, $M_{2,1}^s$ and $B_{2,1}^s$ (cf. [27, 25, 31]):

$$B_{2,1}^{n/2+s} \subset M_{2,1}^s \subset B_{2,1}^s. \tag{1.6}$$

So, comparing $M_{2,1}^s$ with $B_{2,1}^{s+n/2}$, we see that $M_{2,1}^s$ contains a class of functions u satisfying $\|u\|_{M_{2,1}^s} = \infty$ but $\|u\|_{B_{2,1}^{s+n/2}} \ll 1$. On the other hand, we can also find
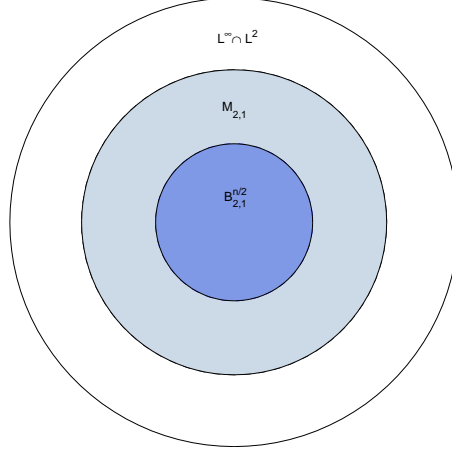
Figure 1: Optimal inclusions: $B_{2,1}^{n/2} \subset M_{2,1} \subset L^\infty \cap L^2$.

a class of rough functions u satisfying $\|u\|_{B_{2,1}^s} = \infty$ but $\|u\|_{M_{2,1}^s} \ll 1$. Another important inclusion between $M_{2,1}$ and L^∞ is that $M_{2,1} \subset L^\infty$ and this embedding is also optimal, see Figure 1.

1.2 Main Results

For the definitions of the anisotropic Lebesgue spaces $L_{x_i}^{p_1} L_{(x_j)_{j\neq i}}^{p_2} L_t^{p_2}(\mathbb{R}^{1+n})$ and the frequency-uniform decomposition operators $\{\square_k\}_{k\in\mathbb{Z}^n}$, one can refer to Section 1.3. We have

Theorem 1.1 *Let $n \geqslant 2$, $2 < m < \infty$, $m > 4/n$. Assume that $u_0 \in M_{2,1}^{3/2}$ and $\|u_0\|_{M_{2,1}^{3/2}} \leq \delta$ for some small $\delta > 0$. Then (1.1) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{3/2}) \cap X$, where*

$$
\begin{aligned}
\|u\|_X = &\sum_{\alpha=0,1} \sum_{i,\ell=1}^n \sum_{k\in\mathbb{Z}^n,\ |k_i|>4} \langle k_i \rangle \left\| \partial_{x_\ell}^\alpha \square_k u \right\|_{L_{x_i}^\infty L_{(x_j)_{j\neq i}}^2 L_t^2(\mathbb{R}^{1+n})} \\
&+ \sum_{\alpha=0,1} \sum_{i,\ell=1}^n \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{1/2-1/m} \left\| \partial_{x_\ell}^\alpha \square_k u \right\|_{L_{x_i}^m L_{(x_j)_{j\neq i}}^\infty L_t^\infty(\mathbb{R}^{1+n})} \\
&+ \sum_{\alpha=0,1} \sum_{\ell=1}^n \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{1/2} \left\| \partial_{x_\ell}^\alpha \square_k u \right\|_{L_t^\infty L_x^2 \cap L_{x,t}^{2+m}(\mathbb{R}^{1+n})}, \quad (1.7)
\end{aligned}
$$

where $k = (k_1, ..., k_n)$ and we further have $\|u\|_X \lesssim \delta$. Moreover, the scattering operator of (1.1) S carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{3/2})$ into $C(\mathbb{R}, M_{2,1}^{3/2})$.

In Theorem 1.1, if $u_0 \in M_{2,1}^s$ with $s > 3/2$, then we have $u \in C(\mathbb{R}, M_{2,1}^s)$. If $m = 2$, we need to assume the initial data have stronger regularity:

Theorem 1.2 *Let $n \geqslant 3$, $m = 2$. Assume that $u_0 \in M_{2,1}^{5/2}$ and $\|u_0\|_{M_{2,1}^{5/2}} \leq \delta$ for some small $\delta > 0$. Then (1.1) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{5/2}) \cap Y$, where*

$$\|u\|_Y = \sum_{\alpha=0,1} \sum_{i,\ell=1}^n \sum_{k \in \mathbb{Z}^n,\ |k_i|>4} \langle k_i \rangle^2 \left\| \partial_{x_\ell}^\alpha \Box_k u \right\|_{L_{x_i}^\infty L_{(x_j)_{j \neq i}}^2 L_t^2 (\mathbb{R}^{1+n})}$$

$$+ \sum_{\alpha=0,1} \sum_{i,\ell=1}^n \sum_{k \in \mathbb{Z}^n} \left\| \partial_{x_\ell}^\alpha \Box_k u \right\|_{L_{x_i}^2 L_{(x_j)_{j \neq i}}^\infty L_t^\infty (\mathbb{R}^{1+n})}$$

$$+ \sum_{\alpha=0,1} \sum_{\ell=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \left\| \partial_{x_\ell}^\alpha \Box_k u \right\|_{L_t^\infty L_x^2 \cap L_t^3 L_x^6 (\mathbb{R}^{1+n})} \quad (1.8)$$

and $\|u\|_Y \lesssim \delta$. Moreover, the scattering operator of (1.1) S carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{5/2})$ into $C(\mathbb{R}, M_{2,1}^{5/2})$.

When the nonlinearity F has a simple form, say,

$$i u_t + \Delta_\pm u = \sum_{i=1}^n \lambda_i \partial_{x_i} (u^{\kappa_i + 1}), \quad u(0, x) = u_0(x), \quad (1.9)$$

we obtained in [31] the global well posedness of the DNLS (1.9) for the small data in modulation spaces $M_{2,1}^{1/\kappa_1}$ in one spatial dimension. In higher spatial dimensions $n \geqslant 2$, we have

Theorem 1.3 *Let $n \geqslant 2$, $\kappa_i > 2$, $\kappa_i > 4/n$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $\kappa = \min_{1 \leq i \leq n} \kappa_i$. Assume that $u_0 \in M_{2,1}^{1/2}$ and $\|u_0\|_{M_{2,1}^{1/2}} \leq \delta$ for some small $\delta > 0$. Then (1.9) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{1/2}) \cap X$, where*

$$\|u\|_{X_1} = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n,\ |k_i|>4} \langle k_i \rangle \left\| \Box_k u \right\|_{L_{x_i}^\infty L_{(x_j)_{j \neq i}}^2 L_t^2 (\mathbb{R}^{1+n})}$$

$$+ \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2 - 1/\kappa} \left\| \Box_k u \right\|_{L_{x_i}^\kappa L_{(x_j)_{j \neq i}}^\infty L_t^\infty (\mathbb{R}^{1+n})}$$

$$+ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \left\| \Box_k u \right\|_{L_t^\infty L_x^2 \cap L_{x,t}^{2+\kappa} (\mathbb{R}^{1+n})} \quad (1.10)$$

and $\|u\|_{X_1} \lesssim \delta$. Moreover, the scattering operator of (1.9) S carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{1/2})$ into $C(\mathbb{R}, M_{2,1}^{1/2})$.

We remark that in Theorem 1.3, the same result holds if the nonlinear term $\partial_{x_i}(u^{\kappa_i+1})$ is replaced by $\partial_{x_i}(|u|^{\kappa_i}u)$ $(\kappa_i \in 2\mathbb{N})$.

Theorem 1.4 *Let $n \geqslant 3$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $\kappa = \min_{1 \leq i \leq n} \kappa_i = 2$. Assume that $u_0 \in M_{2,1}^{3/2}$ and $\|u_0\|_{M_{2,1}^{3/2}} \leq \delta$ for some small $\delta > 0$. Then (1.9) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{3/2}) \cap Y_1$, where*

$$
\begin{aligned}
\|u\|_{Y_1} &= \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n, \ |k_i|>4} \langle k_i \rangle^2 \|\Box_k u\|_{L_{x_i}^\infty L_{(x_j)_{j\neq i}}^2 L_t^2(\mathbb{R}^{1+n})} \\
&\quad + \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \|\Box_k u\|_{L_{x_i}^2 L_{(x_j)_{j\neq i}}^\infty L_t^\infty(\mathbb{R}^{1+n})} \\
&\quad + \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\Box_k u\|_{L_t^\infty L_x^2 \cap L_t^3 L_x^6(\mathbb{R}^{1+n})}.
\end{aligned}
\tag{1.11}
$$

Moreover, the scattering operator of (1.1) S carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{3/2})$ into $C(\mathbb{R}, M_{2,1}^{3/2})$.

Corollary 1.5 *Let $n \geqslant 2$, $m \geqslant 2$, $s > (n+3)/2$. Assume that $u_0 \in H^{s+1}$ and $\|u_0\|_{H^{s+1}} \leq \delta$. Then (1.1) has a unique global solution $u \in X$.*

When $m = 1$, Christ [5] showed the ill posedness of (1.9) in any H^s for one spatial dimension case. For general nonlinearity in (1.1), we do not know what happens in the case $m = 1$ in higher spatial dimensions.

1.3 Notations

The following are some notations which will be frequently used in this paper: $\mathbb{C}, \mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of complex number, reals, positive integers and integers, respectively. $c \leq 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$. We denote by p' the dual number of $p \in [1,\infty]$, i.e., $1/p+1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$, Sobolev spaces $H^s = (I-\Delta)^{-s/2}L^2$. Some properties of these function spaces can be found in [1, 28]. If there is no explanation, we always assume that spatial dimensions $n \geqslant 2$. We will use the function spaces $L_t^q L_x^p(\mathbb{R}^{n+1})$ and $L_x^p L_t^q(\mathbb{R}^{n+1})$ for which the norms are defined by

$$
\|f\|_{L_t^q L_x^p(\mathbb{R}^{n+1})} = \left\| \|f\|_{L_x^p(\mathbb{R}^n)} \right\|_{L_t^q(\mathbb{R})}, \quad \|f\|_{L_x^p L_t^q(\mathbb{R}^{n+1})} = \left\| \|f\|_{L_t^q(\mathbb{R})} \right\|_{L_x^p(\mathbb{R}^n)},
$$

$L_{x,t}^p(\mathbb{R}^{n+1}) := L_x^p L_t^p(\mathbb{R}^{n+1})$. We denote by $L_{x_i}^{p_1} L_{(x_j)_{j\neq i}}^{p_2} L_t^{p_2} := L_{x_i}^{p_1} L_{(x_j)_{j\neq i}}^{p_2} L_t^{p_2}(\mathbb{R}^{1+n})$ the anisotropic Lebesgue space for which the norm is defined by

$$\|f\|_{L_{x_i}^{p_1} L_{(x_j)_{j\neq i}}^{p_2} L_t^{p_2}} = \left\| \|f\|_{L_{x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n}^{p_2} L_t^{p_2}(\mathbb{R}\times\mathbb{R}^{n-1})} \right\|_{L_{x_i}^{p_1}(\mathbb{R})}. \tag{1.12}$$

It is also convenient to use the notation $L_{x_1}^{p_1} L_{x_2,\ldots,x_n}^{p_2} L_t^{p_2} := L_{x_1}^{p_1} L_{(x_j)_{j\neq 1}}^{p_2} L_t^{p_2}$. For any $1 < k < n$, we denote by $\mathscr{F}_{x_1,\ldots,x_k}$ the partial Fourier transform:

$$(\mathscr{F}_{x_1,\ldots,x_k} f)(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_n) = \int_{\mathbb{R}^k} e^{-i(x_1\xi_1 + \ldots + x_k\xi_k)} f(x) dx_1 \ldots dx_k \tag{1.13}$$

and by $\mathscr{F}_{\xi_1,\ldots,\xi_k}^{-1}$ the partial inverse Fourier transform, similarly for $\mathscr{F}_{t,x}$ and $\mathscr{F}_{\tau,\xi}^{-1}$. $\mathscr{F} := \mathscr{F}_{x_1,\ldots,x_n}$, $\mathscr{F}^{-1} := \mathscr{F}_{\xi_1,\ldots,\xi_n}^{-1}$. $D_{x_i}^s = (-\partial_{x_i}^2)^{s/2} = \mathscr{F}_{\xi_i}^{-1} |\xi_i|^s \mathscr{F}_{x_i}$ expresses the partial Riesz potential in the x_i direction. $\partial_{x_i}^{-1} = \mathscr{F}_{\xi_i}^{-1} (i\xi_i)^{-1} \mathscr{F}_{x_i}$. We will use the Bernstein multiplier estimate; cf. [1, 28]. For any $r \in [1, \infty]$,

$$\|\mathscr{F}^{-1} \varphi \mathscr{F} f\|_r \leq C \|\varphi\|_{H^s} \|f\|_r, \quad s > n/2. \tag{1.14}$$

We will use the frequency-uniform decomposition operators (cf. [29, 30, 31]). Let $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ be a function sequence satisfying

$$\begin{cases} \sigma_k(\xi) \geqslant c, & \forall\, \xi \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi : |\xi - k| \leq \sqrt{n}\}, \\ \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1, & \forall\, \xi \in \mathbb{R}^n, \\ |D^\alpha \sigma_k(\xi)| \leq C_m, & \forall\, \xi \in \mathbb{R}^n, |\alpha| \leq m \in \mathbb{N}. \end{cases} \tag{1.15}$$

Denote

$$\Upsilon = \{\{\sigma_k\}_{k\in\mathbb{Z}^n} : \{\sigma_k\}_{k\in\mathbb{Z}^n} \text{ satisfies } (1.15)\}. \tag{1.16}$$

Let $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$ be a function sequence and

$$\Box_k := \mathscr{F}^{-1} \sigma_k \mathscr{F}, \quad k \in \mathbb{Z}^n, \tag{1.17}$$

which are said to be the frequency-uniform decomposition operators. One may ask the existence of the frequency-uniform decomposition operators. Indeed, let $\rho \in \mathscr{S}(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function adapted to the ball $B(0, \sqrt{n})$, say $\rho(\xi) = 1$ as $|\xi| \leq \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \geqslant \sqrt{n}$. Let ρ_k be a translation of ρ: $\rho_k(\xi) = \rho(\xi - k)$, $k \in \mathbb{Z}^n$. We write

$$\eta_k(\xi) = \rho_k(\xi) \left(\sum_{k\in\mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \tag{1.18}$$

We have $\{\eta_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$. It is easy to see that for any $\{\eta_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$,

$$\|f\|_{M_{2,1}^s} \sim \sum_{k\in\mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L^2(\mathbb{R}^n)}.$$

We will use the function space $\ell_{\Box}^{1,s}(L_t^p L_x^r(I\times\mathbb{R}^n))$ which contains all of the functions $f(t,x)$ so that the following norm is finite:

$$\|f\|_{\ell_{\Box}^{1,s}(L_t^p L_x^r(I\times\mathbb{R}^n))} := \sum_{k\in\mathbb{Z}^n} \langle k \rangle^s \|\Box_k f\|_{L_t^p L_x^r(I\times\mathbb{R}^n)}. \tag{1.19}$$

For simplicity, we write $\ell_{\Box}^1(L_t^p L_x^r(I\times\mathbb{R}^n)) = \ell_{\Box}^{1,0}(L_t^p L_x^r(I\times\mathbb{R}^n))$.

This paper is organized as follows. In Section 2 we show the global smooth effect estimates of the solutions of the linear Schrödinger equation in anisotripic Lebesgue spaces. In Sections 3 and 4 we consider the frequency-uniform localized versions for the global maximal function estimates, the global smooth effects, together with their relations to the Strichartz estimates. In Sections 5 and 6 we prove our Theorems 1.3 and 1.1, respectively. In the Appendix we generalize the Christ-Kiselev Lemma to the anisotropic Lebesgue spaces in higher dimensions.

2 Anisotropic global smooth effects

In this section, we always denote

$$S(t) = e^{it\Delta_{\pm}} = \mathscr{F}^{-1} e^{it\sum_{j=1}^n \varepsilon_j \xi_j^2} \mathscr{F}, \quad \mathscr{A}f(t,x) = \int_0^t S(t-\tau)f(\tau,x)d\tau.$$

Proposition 2.1 *For any $i = 1, ..., n$, we have the following estimate:*

$$\|\partial_{x_i}\mathscr{A}f\|_{L_{x_i}^\infty L_{(x_j)_{j\neq i}}^2 L_t^2(\mathbb{R}^{1+n})} \lesssim \|f\|_{L_{x_i}^1 L_{(x_j)_{j\neq i}}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{2.1}$$

Proof. We have

$$\partial_{x_1}\mathscr{A}f = c\mathscr{F}_{t,x}^{-1} \frac{\xi_1}{|\xi|_{\pm}^2 - \tau} \mathscr{F}_{t,x}f. \tag{2.2}$$

We can assume, without loss of generality that $|\xi|_{\pm}^2 = \xi_1^2 + \varepsilon_2\xi_2^2 + ... + \varepsilon_n\xi_n^n := \xi_1^2 + |\bar{\xi}|_{\pm}^2$. By Plancherel's identity,

$$\|\partial_{x_1}\mathscr{A}f\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$= \left\| \mathscr{F}_{\xi_1}^{-1} \frac{\xi_1}{\xi_1^2 + |\bar{\xi}|_{\pm}^2 - \tau} \mathscr{F}_{t,x}f \right\|_{L_{x_1}^\infty L_{\xi_2,...,\xi_n}^2 L_\tau^2(\mathbb{R}^{1+n})}$$

8

$$\leq \left\| \mathscr{F}_{\xi_1}^{-1} \frac{\xi_1}{\xi_1^2 + |\bar{\xi}|_\pm^2 - \tau} \mathscr{F}_{t,x} f \right\|_{L^2_{\xi_2,\ldots,\xi_n} L^\infty_{x_1} L^2_\tau (\mathbb{R}^{1+n})}. \tag{2.3}$$

By changing the variable $\tau \to \mu + |\bar{\xi}|_\pm^2$, we have

$$\left\| \mathscr{F}_{\xi_1}^{-1} \frac{\xi_1}{\xi_1^2 + |\bar{\xi}|_\pm^2 - \tau} \mathscr{F}_{t,x} f \right\|_{L^2_{\xi_2,\ldots,\xi_n} L^\infty_{x_1} L^2_\tau (\mathbb{R}^{1+n})}$$
$$= \left\| \mathscr{F}_{\xi_1}^{-1} \frac{\xi_1}{\xi_1^2 - \mu} \mathscr{F}_{t,x_1} \left(e^{-it|\bar{\xi}|_\pm^2} \mathscr{F}_{x_2,\ldots,x_n} f \right) \right\|_{L^2_{\xi_2,\ldots,\xi_n} L^\infty_{x_1} L^2_\mu (\mathbb{R}^{1+n})}. \tag{2.4}$$

Recalling the smooth effect estimate in one spatial dimension (cf. [13])

$$\left\| \mathscr{F}_{\tau,\xi}^{-1} \frac{\xi}{\xi^2 - \tau} \mathscr{F}_{t,x} f \right\|_{L^\infty_x L^2_t (\mathbb{R}^{1+1})} \lesssim \|f\|_{L^1_x L^2_t (\mathbb{R}^{1+1})}, \tag{2.5}$$

we have from (2.3), (2.4) and (2.5) that

$$\|\partial_{x_1} \mathscr{A} f\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t (\mathbb{R}^{1+n})} \lesssim \left\| e^{-it|\bar{\xi}|_\pm^2} \mathscr{F}_{x_2,\ldots,x_n} f \right\|_{L^2_{\xi_2,\ldots,\xi_n} L^1_{x_1} L^2_t (\mathbb{R}^{1+n})}. \tag{2.6}$$

Using Minkowski's inequality and Plancherel's equality, we immediately have

$$\|\partial_{x_1} \mathscr{A} f\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t (\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1_{x_1} L^2_{\xi_2,\ldots,\xi_n} L^2_t (\mathbb{R}^{1+n})}. \tag{2.7}$$

The other cases can be shown in a similar way. □

Proposition 2.2 *For any $i = 1, \ldots, n$, we have the following estimate:*

$$\left\| D_{x_i}^{1/2} S(t) u_0 \right\|_{L^\infty_{x_i} L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})} \lesssim \|u_0\|_2. \tag{2.8}$$

Proof. By Plancherel's equality and Minkowski's inequality,

$$\|S(t) u_0\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t (\mathbb{R}^{1+n})} = \left\| \mathscr{F}_{\xi_1}^{-1} e^{it\varepsilon_1 \xi_1^2} \mathscr{F}_{x_1} \left(\mathscr{F}_{x_2,\ldots,x_n} u_0 \right) \right\|_{L^\infty_{x_1} L^2_{\xi_2,\ldots,\xi_n} L^2_t (\mathbb{R}^{1+n})}$$
$$\leq \left\| \mathscr{F}_{\xi_1}^{-1} e^{it\varepsilon_1 \xi_1^2} \mathscr{F}_{x_1} \left(\mathscr{F}_{x_2,\ldots,x_n} u_0 \right) \right\|_{L^2_{\xi_2,\ldots,\xi_n} L^\infty_{x_1} L^2_t (\mathbb{R}^{1+n})}. \tag{2.9}$$

Recall the half-order smooth effect of $S(t)$ in one spatial dimension (cf. [13]),

$$\left\| \mathscr{F}_{\xi}^{-1} e^{it\xi^2} \mathscr{F}_x u_0 \right\|_{L^\infty_x L^2_t (\mathbb{R}^{1+1})} \lesssim \|D_x^{-1/2} u_0\|_{L^2(\mathbb{R})}. \tag{2.10}$$

Hence, in view of (2.9) and (2.10), using Plancherel's equality, we immediately have

$$\|S(t) u_0\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t (\mathbb{R}^{1+n})} \lesssim \|D_{x_1}^{-1/2} u_0\|_{L^2(\mathbb{R}^n)}, \tag{2.11}$$

which implies the result, as desired. □

The dual version of (2.8) is

Proposition 2.3 *For any $i = 1, ..., n$, we have the following estimate:*

$$\|\partial_{x_i}\mathscr{A}f\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} \lesssim \|D_{x_i}^{1/2}f\|_{L_{x_i}^1 L_{(x_j)_{j\neq i}}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{2.12}$$

Proof. Denote $\mathbb{R}_+ = [0, \infty)$. By Proposition 2.2,

$$\left| \int_{\mathbb{R}_+} ((\mathscr{A}\partial_{x_1}f)(t),\ \psi(t))\, dt \right|$$

$$= \left| \int_{\mathbb{R}_+} \left(f(\tau),\ \int_\tau^\infty S(\tau - t)\partial_{x_1}\psi(t)dt \right) d\tau \right|$$

$$\leq \|D_{x_1}^{1/2}f\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})} \int_{\mathbb{R}_+} \|\partial_{x_1}S(\tau - t)D_{x_1}^{-1/2}\psi(t)\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_\tau^2(\mathbb{R}^{1+n})} dt$$

$$\lesssim \|D_{x_1}^{1/2}f\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}\|\psi\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})}. \tag{2.13}$$

By duality, we have the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3   Linear estimates with $\square_k$-decomposition

In this section we consider the smooth effect estimates, the maximal function estimates, the Strichartz estimates and their interaction estimates for the solutions of the linear Schrödinger equations by using the frequency-uniform decomposition operators. For convenience, we will use the following function sequence $\{\sigma_k\}_{k\in\mathbb{Z}^n}$:

Lemma 3.1 *Let $\eta_k : \mathbb{R} \to [0, 1]$ $(k \in \mathbb{Z})$ be a smooth-function sequence satisfying condition* (1.15). *Denote*

$$\sigma_k(\xi) := \eta_{k_1}(\xi_1)...\eta_{k_n}(\xi_n), \quad k = (k_1, ..., k_n). \tag{3.1}$$

Then we have $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$.

Recall that in [30], we established the following Strichartz estimates in a class of function spaces by using the frequency-uniform decomposition operators.

Lemma 3.2 *Let $2 \leq p < \infty$, $\gamma \geqslant 2 \vee \gamma(p)$,*

$$\frac{2}{\gamma(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right).$$

Then we have

$$\|S(t)\varphi\|_{\ell_\square^1(L^\gamma(\mathbb{R}, L^p(\mathbb{R}^n)))} \lesssim \|\varphi\|_{M_{2,1}(\mathbb{R}^n)},$$

10

$$\|\mathscr{A}f\|_{\ell^1_\square(L^\gamma(\mathbb{R},L^p(\mathbb{R}^n)))\cap\ell^1_\square(L^\infty(\mathbb{R},L^2(\mathbb{R}^n)))} \lesssim \|f\|_{\ell^1_\square(L^{\gamma'}(\mathbb{R},L^{p'}(\mathbb{R}^n)))}.$$

In particular, if $2 + 4/n \leq p < \infty$, then we have

$$\|S(t)\varphi\|_{\ell^1_\square(L^p_{t,x}(\mathbb{R}^{1+n}))} \lesssim \|\varphi\|_{M_{2,1}(\mathbb{R}^n)},$$

$$\|\mathscr{A}f\|_{\ell^1_\square(L^p_{t,x}(\mathbb{R}^{1+n}))\cap\ell^1_\square(L^\infty_t L^2_x(\mathbb{R}^{1+n}))} \lesssim \|f\|_{\ell^1_\square(L^{p'}_{t,x}(\mathbb{R}^{1+n}))}.$$

The next lemma is essentially known, see [28, 29].

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a compact set with $\operatorname{diam}\Omega < 2R$, $0 < p \leq q \leq \infty$. Then there exists a constant $C > 0$, which depends only on p, q such that*

$$\|f\|_q \leq CR^{n(1/p-1/q)}\|f\|_p, \quad \forall\, f \in L^p_\Omega,$$

where $L^p_\Omega = \{f \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp}\hat{f} \subset \Omega,\ \|f\|_p < \infty\}$.

In Lemma 3.3 we emphasize that the constant $C > 0$ is independent of the position of Ω in frequency spaces, say, in the case $\Omega = B(k, \sqrt{n})$, $k \in \mathbb{Z}^n$, Lemma 3.3 uniformly holds for all $k \in \mathbb{Z}^n$.

Lemma 3.4 *We have for any $\sigma \in \mathbb{R}$ and $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ with $|k_i| \geqslant 4$,*

$$\|\square_k D^\sigma_{x_i} u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,...,x_n} L^{p_2}_t(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^\sigma \|\square_k u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,...,x_n} L^{p_2}_t(\mathbb{R}^{1+n})}.$$

Replacing $D^\sigma_{x_i}$ by $\partial^\sigma_{x_i}$ $(\sigma \in \mathbb{N})$, the above inequality holds for all $k \in \mathbb{Z}^n$.

Proof. Using Lemma 3.1, one has that

$$\square_k D^\sigma_{x_i} u = \sum_{\ell=-1}^{1} \int_{\mathbb{R}} \left(\mathscr{F}^{-1}_{\xi_i}(\eta_{k_i+\ell}(\xi_i)|\xi_i|^\sigma)\right)(y_i)(\square_k u)(x_i - y_i) dy_i.$$

It follows that

$$\|\square_k D^\sigma_{x_i} u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,...,x_n} L^{p_2}_t(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{\ell=-1}^{1} \|\mathscr{F}^{-1}_{\xi_i}(\eta_{k_i+\ell}(\xi_i)|\xi_i|^\sigma)\|_{L^1(\mathbb{R})} \|\square_k u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,...,x_n} L^{p_2}_t(\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_i \rangle^\sigma \|\square_k u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,...,x_n} L^{p_2}_t(\mathbb{R}^{1+n})}.$$

The result follows. $\square$

Ionescu and Kenig [12] showed the following maximal function estimates in higher spatial dimensions $n \geqslant 3$:

$$\|\triangle_k S(t)u_0\|_{L^2_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})} \lesssim 2^{(n-1)k/2}\|\triangle_k u_0\|_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

We partially resort to their idea to obtain the following

Proposition 3.5 *Let* $4/n < q \le \infty$, $q \ge 2$. *Then we have*

$$\|\Box_k S(t)u_0\|_{L^q_{x_i} L^\infty_{(x_j)_{j\ne i}} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/q} \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}. \tag{3.3}$$

Proof. For convenience, we write $\bar{x} = (x_1, ..., x_{n-1})$. By duality, it suffices to show that for any $\varphi \in L^{q'}_{x_1} L^1_{\bar{x},t}(\mathbb{R}^{1+n}) \bigcap \mathscr{S}(\mathbb{R}^{1+n})$ with $\varphi(t) = \pm\varphi(-t)$,

$$\int_{\mathbb{R}} (\Box_k S(t)u_0, \ \varphi(t))dt \lesssim \langle k_i \rangle^{1/q} \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \|S(t)\varphi\|_{L^{q'}_{x_1} L^1_{\bar{x},t}(\mathbb{R}^{1+n})}. \tag{3.4}$$

By duality, we have

$$\int_{\mathbb{R}} (\Box_k S(t)u_0, \ \varphi(t))dt \lesssim \|u_0\|_{L^2(\mathbb{R}^n)} \left\| \int_{\mathbb{R}} \Box_k S(-t)\varphi(t)dt \right\|_{L^2(\mathbb{R}^n)}. \tag{3.5}$$

We have from Lemma 3.4 that

$$\left\| \int_{\mathbb{R}} \Box_k S(-t)\varphi(t)dt \right\|^2_{L^2(\mathbb{R}^n)} \lesssim \|S(t)\varphi\|_{L^{q'}_{x_1} L^1_{\bar{x},t}(\mathbb{R}^{1+n})} \left\| \int_{\mathbb{R}} \Box_k S(2t - \tau)\varphi(\tau)d\tau \right\|_{L^q_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^{1+n})}. \tag{3.6}$$

In view of Lemma 3.1, we can write $\Box_k = \mathscr{F}^{-1}\eta_{k_1}(\xi_1)...\eta_{k_n}(\xi_n)\mathscr{F} := \mathscr{F}^{-1}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\mathscr{F}$. By Minkowski's and Young's inequalities,

$$\left\| \int_{\mathbb{R}} \Box_k S(2t - \tau)\varphi(\tau)d\tau \right\|_{L^q_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^{1+n})}$$

$$\lesssim \left\| \mathscr{F}^{-1}e^{i2t|\xi|^2_\pm}\eta_{\bar{k}}(\bar{\xi})\eta_{k_1}(\xi_1)\mathscr{F} \int_{\mathbb{R}} S(-\tau)\varphi(\tau)d\tau \right\|_{L^q_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^{n+1})}$$

$$\lesssim \left\| \|\mathscr{F}^{-1}_{\bar{\xi}}e^{i2t|\bar{\xi}|^2_\pm}\eta_{\bar{k}}(\bar{\xi})\|_{L^\infty_{\bar{x}}(\mathbb{R}^n)} \left\| \mathscr{F}^{-1}_{\xi_1}e^{i2t\xi_1^2}\eta_{k_1}(\xi_1)\mathscr{F}_{x_1} \int_{\mathbb{R}} S(-\tau)\varphi(\tau)d\tau \right\|_{L^1_{\bar{x}}} \right\|_{L^q_{x_1} L^\infty_t}$$

$$\lesssim \|\mathscr{F}^{-1}e^{i2t|\xi|^2_\pm}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\|_{L^{q/2}_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^{n+1})} \left\| \int_{\mathbb{R}} S(-\tau)\varphi(\tau)d\tau \right\|_{L^{q'}_{x_1} L^1_{\bar{x}}(\mathbb{R}^n)}$$

$$\lesssim \|\mathscr{F}^{-1}e^{i2t|\xi|^2_\pm}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\|_{L^{q/2}_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^{1+n})} \|S(\tau)\varphi\|_{L^{q'}_{x_1} L^1_{\bar{x},t}(\mathbb{R}^{n+1})}. \tag{3.7}$$

Hence, it suffices to show that

$$\|\mathscr{F}^{-1}e^{it|\xi|^2_\pm}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\|_{L^{q/2}_{x_1} L^\infty_{\bar{x},t}(\mathbb{R}^n)} \lesssim \langle k_1 \rangle^{2/q}.$$

In view of the decay of $\Box_k S(t)$, we see that (cf. [30])

$$\|\mathscr{F}^{-1}_{\bar{\xi}}e^{it|\bar{\xi}|^2_\pm}\eta_{\bar{k}}(\bar{\xi})\|_{L^\infty_{\bar{x}}(\mathbb{R}^{n-1})} \lesssim (1 + |t|)^{-(n-1)/2},$$

$$\|\mathscr{F}_{\xi_1}^{-1} e^{it\xi_1^2} \eta_{k_1}(\xi_1)\|_{L_{x_1}^\infty(\mathbb{R})} \lesssim (1+|t|)^{-1/2}.$$

On the other hand, integrating by part, one has that for $|x_1| > 4|t|\langle k_1 \rangle$,

$$|\mathscr{F}_{\xi_1}^{-1} e^{it\xi_1^2} \eta_{k_1}(\xi_1)| \lesssim |x_1|^{-2}.$$

Hence, for $|x_1| > 1$,

$$|\mathscr{F}^{-1} e^{it|\xi|_\pm^2} \eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})| \lesssim (1+|x_1|)^{-2} + \langle k_1 \rangle^{n/2} (\langle k_1 \rangle + |x_1|)^{-n/2}.$$

So, we have

$$\|\mathscr{F}^{-1} e^{it|\xi|_\pm^2} \eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\|_{L_{x_1}^{q/2} L_{\bar{x},t}^\infty(\mathbb{R}^n)} \lesssim 1 + \langle k_1 \rangle^{n/2} \|(\langle k_1 \rangle + |x_1|)^{-n/2}\|_{L_{x_1}^{q/2}(\mathbb{R})}$$
$$\lesssim \langle k_1 \rangle^{2/q}.$$

This finishes the proof of (3.3). □

Remark 3.6 We conjecture that (3.3) also holds in the case $p = 4/n$ if $n = 2$. We now show that (3.3) is sharp. Indeed, take $k_1 \in \mathbb{N}$, $\mathscr{F}_x u_0(\xi) = \eta_{k_1}(\xi_1)\eta_0(\xi_2)...\eta_0(\xi_n)$, where η_k is as in Lemma 3.1. For $\sigma_k := \eta_{k_1}(\xi_1)\eta_0(\xi_2)...\eta_0(\xi_n)$, we easily see that

$$\|\Box_k S(t)u_0\|_{L_{x_1}^q L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}^q$$
$$= \left\| \mathscr{F}_{\xi_1}^{-1}(e^{i\varepsilon_1 t\xi_1^2} \eta_{k_1}^2(\xi_1)) \prod_{i=2}^n \mathscr{F}_{\xi_i}^{-1}(e^{i\varepsilon_i t\xi_i^2} \eta_0^2(\xi_i)) \right\|_{L_{x_1}^q L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}^q$$
$$\geqslant \int_{|x_1| \leq ck_1} \sup_{t, x_2,...,x_n} \left| \mathscr{F}_{\xi_1}^{-1}(e^{i\varepsilon_1 t\xi_1^2} \eta_{k_1}^2(\xi_1))(x_1) \prod_{i=2}^n \mathscr{F}_{\xi_i}^{-1}(e^{i\varepsilon_i t\xi_i^2} \eta_0^2(\xi_i))(x_i) \right|^q dx_1.$$

Taking $t = -x_1/2\varepsilon_1 k_1$ and $|x_i| < c$, $i = 2,...,n$, we easily see that

$$\left| \mathscr{F}_{\xi_1}^{-1}(e^{i\varepsilon_1 t\xi_1^2} \eta_{k_1}^2(\xi_1))(x_1) \right| \gtrsim 1,$$
$$\left| \mathscr{F}_{\xi_i}^{-1}(e^{i\varepsilon_i t\xi_i^2} \eta_0^2(\xi_i))(x_i) \right| \gtrsim 1.$$

Therefore, we have

$$\|\Box_k S(t)u_0\|_{L_{x_1}^q L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}^q \gtrsim k_1.$$

The dual version of Proposition 3.5 is the following

Proposition 3.7 Let $2 \leq q \leq \infty$, $q > 4/n$. Then we have for any $k = (k_1,...,k_n) \in \mathbb{Z}^n$,

$$\left\| \Box_k \int_{\mathbb{R}} S(t-\tau)f(\tau)d\tau \right\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/q} \|\Box_k f\|_{L_{x_i}^{q'} L_{(x_j)_{j \neq i}}^1 L_t^1(\mathbb{R}^{1+n})}. \tag{3.8}$$

13

Proof. Denote

$$\tilde{\Box}_k = \sum_{\ell \in \Lambda} \Box_{k+\ell}, \quad \Lambda = \{\ell \in \mathbb{Z}^n : \operatorname{supp} \sigma_k \cap \operatorname{supp} \sigma_{k+\ell} \neq \varnothing\}. \tag{3.9}$$

Write

$$\mathcal{L}_k(f, \psi) := \left| \int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} S(t-\tau) f(\tau) d\tau, \ \psi(t) \right) dt \right| \tag{3.10}$$

By Proposition 3.5,

$$\mathcal{L}_k(f, \psi) = \left| \left(\Box_k f(\tau), \ \tilde{\Box}_k \int_{\mathbb{R}} S(\tau - t) \psi(t) dt \right) d\tau \right|$$

$$\leq \|\Box_k f\|_{L_{x_i}^{q'} L^1_{(x_j)_{j \neq i}} L^1_t (\mathbb{R}^{1+n})} \left\| \tilde{\Box}_k \int_{\mathbb{R}} S(\tau - t) \psi(t) dt \right\|_{L_{x_i}^q L^\infty_{(x_j)_{j \neq i}} L^\infty_t (\mathbb{R}^{1+n})}$$

$$\leq \|\Box_k f\|_{L_{x_i}^{q'} L^1_{(x_j)_{j \neq i}} L^1_t (\mathbb{R}^{1+n})} \langle k_i \rangle^{1/q} \|\tilde{\Box}_k \psi\|_{L^1_t L^2_x (\mathbb{R}^{1+n})}. \tag{3.11}$$

By duality, we have the result, as desired. $\qquad \square$

In view of Propositions 2.1 and 2.3, we have

Proposition 3.8 *We have for any* $k = (k_1, ..., k_n) \in \mathbb{Z}^n$,

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L_{x_i}^\infty L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L_{x_i}^1 L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})}, \tag{3.12}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L_t^\infty L^2_x (\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2} \|\Box_k f\|_{L_{x_i}^1 L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})}. \tag{3.13}$$

Proof. By Proposition 2.1, we immediately have (3.12). In view of Proposition 2.3 and Lemma 3.4, we have (3.13) in the case $|k_i| \geqslant 3$. If $|k_i| \leq 2$, in view of Proposition 2.3,

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L_t^\infty L^2_x (\mathbb{R}^{1+n})} \lesssim \|D_{x_i}^{-1/2} \Box_k \mathscr{A} \partial_{x_i} f\|_{L_t^\infty L^2_x (\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L_{x_i}^1 L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})},$$

which implies the result, as desired. $\qquad \square$

By the duality, we also have the following

Proposition 3.9 *Let* $2 < q \leq \infty$ $q > 4/n$. *Then we have*

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L_{x_i}^q L^\infty_{(x_j)_{j \neq i}} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2 + 1/q} \|\Box_k f\|_{L_{x_i}^1 L^2_{(x_j)_{j \neq i}} L^2_t (\mathbb{R}^{1+n})}. \tag{3.14}$$

Proof. By Propositions 3.7, 3.8 and Lemma 3.4,

$$\mathcal{L}_k(\partial_{x_1} f, \psi) = \left| \left(\Box_k \int_{\mathbb{R}} S(-\tau) \partial_{x_1} f(\tau) d\tau, \ \tilde{\Box}_k \int_{\mathbb{R}} S(-t) \psi(t) dt \right) \right|$$

14

$$\leq \left\|\Box_k \int_{\mathbb{R}} S(-\tau)\partial_{x_1}f(\tau)d\tau\right\|_{L^2(\mathbb{R}^n)} \left\|\tilde{\Box}_k \int_{\mathbb{R}} S(-t)\psi(t)dt\right\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\ldots,x_n}L^2_t(\mathbb{R}^{1+n})}\langle k_1\rangle^{1/q}\left\|\tilde{\Box}_k\psi\right\|_{L^{q'}_{x_1}L^1_{x_2,\ldots,x_n}L^1_t(\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_1\rangle^{1/2+1/q}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\ldots,x_n}L^2_t(\mathbb{R}^{1+n})}\|\psi\|_{L^{q'}_{x_1}L^1_{x_2,\ldots,x_n}L^1_t(\mathbb{R}^{1+n})}. \tag{3.15}$$

Again, by duality, it follows from (3.15) and Christ-Kiselev's Lemma that (3.14) holds. $\qquad\square$

Proposition 3.10 *Let* $2 \leq r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$ *and* $\gamma > \gamma(r) \vee 2$. *We have*

$$\|\Box_k S(t)u_0\|_{L^\gamma_t L^r_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.16}$$

$$\|\Box_k \mathscr{A} f\|_{L^\infty_t L^2_x \cap L^\gamma_t L^r_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}, \tag{3.17}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\gamma_t L^r_x(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2}\|\Box_k f\|_{L^1_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})}, \tag{3.18}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\infty_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2}\|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}, \tag{3.19}$$

and for $2 \leq q < \infty$, $q > 4/n$, $\alpha = 0, 1$,

$$\left\|\Box_k \mathscr{A} \partial_{x_i}^\alpha f\right\|_{L^q_{x_i}L^\infty_{(x_j)_{j\neq i}}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{\alpha+1/q}\|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}, \tag{3.20}$$

Proof. From Lemma 3.2 it follows that (3.16) and (3.17) hold. We now show (3.18). We use the same notations as in Proposition 3.9. By Lemmas 3.2, 3.4 and Proposition 3.8,

$$\mathcal{L}_k(\partial_{x_1}f, \psi) \lesssim \langle k_i\rangle^{1/2}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\ldots,x_n}L^2_t(\mathbb{R}^{1+n})}\left\|\tilde{\Box}_k\psi\right\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_i\rangle^{1/2}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\ldots,x_n}L^2_t(\mathbb{R}^{1+n})}\|\psi\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}. \tag{3.21}$$

By duality, it follows from (3.28) and Christ-Kiselev's Lemma that (3.18) holds. Exchanging the roles of f and ψ, we immediately have (3.19) in the case $r > 2$. If $r = 2$, (3.19) is a straightforward consequence of the $1/2$-order smooth effect of $S(t)$. By Lemmas 3.2, 3.4, Proposition 3.7, and Christ-Kiselev's Lemma that we have (3.20) in the case $q > 2$, or $q = 2$ and $r > 2$. In the case $q = r = 2$, in view of the maximal function estimate, we see that (3.20) also holds. $\qquad\square$

Corollary 3.11 *Let* $4/n \leq p < \infty$, $2 \leq q < \infty$, $q > 4/n$. *We have*

$$\left\|D_{x_1}^{1/2}\Box_k S(t)u_0\right\|_{L^\infty_{x_1}L^2_{x_2,\ldots,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.22}$$

15

$$\|\Box_k S(t)u_0\|_{L^q_{x_1}L^\infty_{x_2,...,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/q}\|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.23}$$

$$\|\Box_k S(t)u_0\|_{L^{2+p}_{t,x}\cap L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.24}$$

$$\|\Box_k \mathscr{A}\partial_{x_1}f\|_{L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^1_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})}, \tag{3.25}$$

$$\|\Box_k \mathscr{A}\partial_{x_1}f\|_{L^q_{x_1}L^\infty_{x_2,...,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2+1/q}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})}, \tag{3.26}$$

$$\|\Box_k \mathscr{A}f\|_{L^\infty_t L^2_x\cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{3.27}$$

$$\|\Box_k \mathscr{A}\partial_{x_1}f\|_{L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}, \tag{3.28}$$

$$\|\Box_k \mathscr{A}\partial_{x_1}f\|_{L^q_{x_1}L^\infty_{x_2,...,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1+1/q}\|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}, \tag{3.29}$$

$$\|\Box_k \mathscr{A}f\|_{L^\infty_t L^2_x\cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}. \tag{3.30}$$

In (3.26), $q > 2$ is required. Moreover, replacing $L^{2+p}_{x,t}$ by $L^3_t L^6_x$, the results also hold.

4 Linear estimates with derivative interaction

In view of (3.25) in Corollary 3.11, the operator $\mathscr{A}$ in the space $L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})$ has succeed in absorbing the partial derivative ∂_{x_1}. However, it seem that $\mathscr{A}$ can not deal with the partial derivative ∂_{x_2} in the space $L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})$. So, we need a new way to handle the interaction between $L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})$ and ∂_{x_2}. We have the following

Proposition 4.1 *Let* $i = 2,...,n$, $2 \le q \le \infty$, $q > 4/n$. *Let* $2 \le r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$, $\gamma \geqslant \gamma(r)$, $\gamma > 2$. *Then we have*

$$\|\Box_k \partial_{x_i}\mathscr{A}f\|_{L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \|\partial_{x_i}\partial_{x_1}^{-1}\Box_k f\|_{L^1_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})}, \tag{4.1}$$

$$\|\Box_k \partial_{x_i}\mathscr{A}f\|_{L^\infty_{x_1}L^2_{x_2,...,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \|\partial_{x_i}D_{x_1}^{-1/2}\Box_k f\|_{L^{\gamma'}L^{r'}_x(\mathbb{R}^{1+n})}, \tag{4.2}$$

$$\|\Box_k \partial_{x_i}\mathscr{A}f\|_{L^q_{x_1}L^\infty_{x_2,...,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2}\langle k_1\rangle^{1/q}\|\Box_k f\|_{L^1_{x_i}L^2_{(x_j)_{j\ne i}}L^2_t(\mathbb{R}^{1+n})}, \tag{4.3}$$

$$\|\Box_k \partial_{x_i}\mathscr{A}f\|_{L^q_{x_1}L^\infty_{x_2,...,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle\langle k_1\rangle^{1/q}\|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}. \tag{4.4}$$

In (4.3), $q > 2$ is required.

Proof. (4.1) is a straightforward consequence of Proposition 2.1. We have

$$\mathcal{L}(\partial_{x_2}f, \psi) := \left|\int_\mathbb{R}\left(\int_\mathbb{R} S(t-\tau)\partial_{x_2}f(\tau)d\tau, \ \psi(t)\right)dt\right|$$

16

$$\leq \left\| \int_{\mathbb{R}} S(-\tau)\partial_{x_2}D_{x_1}^{-1/2}f(\tau)d\tau \right\|_{L^2(\mathbb{R}^n)} \left\| D_{x_1}^{1/2}\int_{\mathbb{R}} S(-t)\psi(t)dt \right\|_{L^2(\mathbb{R}^n)}. \quad (4.5)$$

By the Strichartz inequality and Proposition 2.3,

$$\mathcal{L}(\partial_{x_2}f,\psi) \lesssim \|\partial_{x_2}D_{x_1}^{-1/2}f\|_{L_t^{\gamma'}L_x^{r'}(\mathbb{R}^{1+n})}\|\psi\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \quad (4.6)$$

By duality, (4.6) implies (4.2) in the case $r > 2$. In the case $r = 2$, in view of the $1/2$-order smooth effect of $S(t)$, we see that (4.2) also holds true. Similarly, in view of Propositions 2.3, 3.7 and Lemma 3.4,

$$\mathcal{L}(\partial_{x_2}\Box_k f,\psi) \leq \left\| \int_{\mathbb{R}} S(-\tau)D_{x_2}^1\Box_k f(\tau)d\tau \right\|_{L^2(\mathbb{R}^n)} \left\| \tilde{\Box}_k \int_{\mathbb{R}} S(-t)\psi(t)dt \right\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim \langle k_2 \rangle^{1/2}\|\Box_k f\|_{L_{x_2}^1 L_{(x_j)_{j\neq2}}^2 L_t^2(\mathbb{R}^{1+n})}\langle k_1 \rangle^{1/q}\|\tilde{\Box}_k \psi\|_{L_{x_1}^{q'} L_{x_2,\dots,x_n}^1 L_t^1(\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_2 \rangle^{1/2}\langle k_1 \rangle^{1/q}\|\Box_k f\|_{L_{x_2}^1 L_{(x_j)_{j\neq2}}^2 L_t^2(\mathbb{R}^{1+n})}\|\psi\|_{L_{x_1}^{q'} L_{x_2,\dots,x_n}^1 L_t^1(\mathbb{R}^{1+n})}. \quad (4.7)$$

By duality, (4.3) follows from (4.7). Finally,

$$\mathcal{L}(\partial_{x_2}\Box_k f,\psi) \leq \left\| \int_{\mathbb{R}} S(-\tau)\partial_{x_2}\Box_k f(\tau)d\tau \right\|_{L^2(\mathbb{R}^n)} \left\| \tilde{\Box}_k \int_{\mathbb{R}} S(-t)\psi(t)dt \right\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim \langle k_2 \rangle\langle k_1 \rangle^{1/q}\|\psi\|_{L_{x_1}^{q'} L_{x_2,\dots,x_n}^1 L_t^1(\mathbb{R}^{1+n})}\|\Box_k f\|_{L_t^{\gamma'}L_x^{r'}(\mathbb{R}^{1+n})}. \quad (4.8)$$

If $r > 2$ or $q > 2$, (4.8) and Christ-Kiselev's Lemma imply (7.2), as desired. If $r = q = 2$, in view of Proposition 3.5, we have also (7.2). $\qquad\square$

Lemma 4.2 *Let* $\psi : [0,\infty) \to [0,1]$ *be a smooth bump function satisfying* $\psi(x) = 1$ *as* $|x| \leq 1$ *and* $\psi(x) = 0$ *if* $|x| \geqslant 2$*. Denote* $\psi_1(\xi) = \psi(\xi_2/2\xi_1)$, $\psi_2(\xi) = 1-\psi(\xi_2/2\xi_1)$, $\xi \in \mathbb{R}^n$*. Then we have for* $\sigma \geqslant 0$,

$$\sum_{k\in\mathbb{Z}^n, |k_1|>4} \langle k_1 \rangle^\sigma \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1}\psi_1\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f \right\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{k\in\mathbb{Z}^n, |k_1|>4} \langle k_1 \rangle^\sigma \|\Box_k f\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}, \quad (4.9)$$

and for $\sigma \geqslant 1$,

$$\sum_{k\in\mathbb{Z}^n, |k_1|>4} \langle k_1 \rangle^\sigma \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1}\psi_2\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f \right\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{k\in\mathbb{Z}^n, |k_2|>4} \langle k_2 \rangle^\sigma \|\Box_k f\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \quad (4.10)$$

17

Proof. For simplicity, we denote

$$I = \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} \psi_1 \mathscr{F}_{x_1,x_2} \Box_k \partial_{x_2} \mathscr{A} f \right\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})},$$

$$II = \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} \psi_2 \mathscr{F}_{x_1,x_2} \Box_k \partial_{x_2} \mathscr{A} f \right\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}.$$

Let η_k be as in Lemma 3.1. For $k \in \mathbb{Z}^n$, $|k_1| > 4$, applying the almost orthogonality of $\Box_k$, we have

$$I \lesssim \sum_{|\ell_1|,|\ell_2| \leq 1} \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} \psi\left(\frac{\xi_2}{2\xi_1}\right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \mathscr{F}_{x_1,x_2} \Box_k \partial_{x_1} \mathscr{A} f \right\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{4.11}$$

Denote

$$(f \circledast_{12} g)(x) = \int_{\mathbb{R}^2} f(t, x_1 - y_1, x_2 - y_2, x_3, \dots, x_n) g(t, y_1, y_2) dy_1 dy_2. \tag{4.12}$$

We have for any Banach function space X defined on $\mathbb{R}^{1+n}$,

$$\|f \circledast_{12} g\|_X \leq \|g\|_{L_{y_1,y_2}^1(\mathbb{R}^2)} \sup_{y_1,y_2} \|f(\cdot, \cdot - y_1, \cdot - y_2, \cdot, \dots, \cdot)\|_X. \tag{4.13}$$

Hence, by (4.11) and (4.13),

$$I \lesssim \sum_{|\ell_1|,|\ell_2| \leq 1} \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} \psi\left(\frac{\xi_2}{2\xi_1}\right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)} \|\Box_k \partial_{x_1} \mathscr{A} f\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{4.14}$$

Using Bernstein's multiplier estimate, for $|k_1| > 4$, we have

$$\left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} \psi\left(\frac{\xi_2}{2\xi_1}\right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)}$$

$$\lesssim \sum_{|\alpha| \leq 2} \left\| D^\alpha \left[\psi\left(\frac{\xi_2}{2\xi_1}\right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right] \right\|_{L^2(\mathbb{R}^2)} \lesssim 1. \tag{4.15}$$

By Proposition 3.8, (4.14) and (4.15), we have

$$I \lesssim \|\Box_k f\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}, \quad |k_1| \geqslant 4. \tag{4.16}$$

Next, we consider the estimate of II. Using Proposition 4.1,

$$II \lesssim \left\| \mathscr{F}_{\xi_1,\xi_2}^{-1} (\xi_2/\xi_1) \psi_2 \mathscr{F}_{x_1,x_2} \Box_k f \right\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{|\ell_1|,|\ell_2|\leq 1} \left\| \mathscr{F}^{-1}_{\xi_1,\xi_2} \left(1 - \psi\left(\frac{\xi_2}{2\xi_1}\right) \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)}$$

$$\times \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}. \tag{4.17}$$

Notice that supp$\psi_2 \subset \{\xi : |\xi_2| \geqslant 2|\xi_1|\}$. If $|k_1| \geqslant 4$, we have $|k_2| > 6$ and $|k_2| \geqslant |k_1|$ in the summation of the left-hand side of (4.10). So, $\sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle^\sigma II \leq \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_2\rangle^{\sigma-1} \langle k_1\rangle II.$

$$\left\| \mathscr{F}^{-1}_{\xi_1,\xi_2} \left(1 - \psi\left(\frac{\xi_2}{2\xi_1}\right) \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right\|_{L^1(\mathbb{R}^2)}$$

$$\lesssim \sum_{|\alpha|\leq 2} \left\| D^\alpha \left[\mathscr{F}^{-1}_{\xi_1,\xi_2} \left(1 - \psi\left(\frac{\xi_2}{2\xi_1}\right) \right) \frac{\xi_2}{\xi_1} \prod_{i=1,2} \eta_{k_i+\ell_i}(\xi_i) \right] \right\|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \langle k_2\rangle \langle k_1\rangle^{-1}. \tag{4.18}$$

(4.17) and (4.18) yield the estimate of II, as desired. $\qquad\square$

Conjecture 4.3 Using a similar way as in the proof of (4.1), we can show that

$$\left\| \Box_k \partial_{x_2} \int_{\mathbb{R}} S(t-\tau)f(\tau)d\tau \right\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \|D^{1/2}_{x_2} D^{-1/2}_{x_1} \Box_k f\|_{L^1_{x_2} L^2_{x_1,x_3,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}.$$

So, we can conjecture that

$$\|\Box_k \partial_{x_2} \mathscr{A} f\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \|D^{1/2}_{x_2} D^{-1/2}_{x_1} \Box_k f\|_{L^1_{x_2} L^2_{x_1,x_3,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}. \tag{4.19}$$

Since we do not know if the Christ-Kiselev Lemma holds in the endpoint case, it is not clear for us if (4.19) is true.

If (4.19) is true, repeating the proof above, we can show that (4.9) holds for all $\sigma \geqslant 1/2$. We can improve the results of Theorems 1.1 and 1.3 by assuming that $u_0 \in M^{1+1/m}_{2,1}$ and $u_0 \in M^{1/m}_{2,1}$, respectively.

Lemma 4.4 *Let* $2 \leq q \leq \infty$, $q > 4/n$ *and* $(\gamma,\ r)$ *be as in Proposition 4.1. Let* $k = (k_1, \ldots, k_n)$, $k_{\max} := \max_{1\leq i\leq n} |k_i|$. *Then we have*

$$\|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_{\max}\rangle^{1+1/q} \|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}. \tag{4.20}$$

Proof. It follows from (7.2) that (4.20) holds. $\qquad\square$

Lemma 4.5 *Let $k = (k_1, ..., k_n)$, $k_{\max} := \max_{1 \leq i \leq n} |k_i|$ and $q > 2 \vee 4/n$. Then we have for $\sigma \geqslant 0$ and $i, \alpha = 1, ..., n$,*

$$\sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\max} > 4} \langle k \rangle^\sigma \left\| \Box_k \partial_{x_i} \mathscr{A} f \right\|_{L^q_{x_1} L^\infty_{x_2,...,x_n} L^\infty_t (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{k \in \mathbb{Z}^n, |k_\alpha| > 4} \langle k_\alpha \rangle^{\sigma+1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_\alpha} L^2_{(x_j)_{j \neq \alpha}} L^2_t (\mathbb{R}^{1+n})}. \tag{4.21}$$

Proof. First, we consider the case $\alpha = 1$. In view of (3.26) and $|k_1| = k_{\max} > 4$,

$$\left\| \Box_k \partial_{x_i} \mathscr{A} f \right\|_{L^q_{x_1} L^\infty_{x_2,...,x_n} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \sum_{|\ell_1|, |\ell_i| \leq 1} \left\| \mathscr{F}^{-1}_{\xi_1, \xi_i} \left(\frac{\xi_i}{\xi_1} \eta_{k_i + \ell_i}(\xi_i) \eta_{k_1 + \ell_1}(\xi_1) \right) \right\|_{L^1(\mathbb{R}^2)}$$

$$\times \left\| \Box_k \partial_{x_1} \mathscr{A} f \right\|_{L^q_{x_1} L^\infty_{x_2,...,x_n} L^\infty_t (\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_i \rangle \langle k_1 \rangle^{-1} \langle k_1 \rangle^{1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_1} L^2_{x_2,...,x_n} L^2_t (\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_1 \rangle^{1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_1} L^2_{x_2,...,x_n} L^2_t (\mathbb{R}^{1+n})}. \tag{4.22}$$

(4.22) implies the result, as desired. Next, we consider the case $\alpha = 2$. Notice that $|k_2| = \max_{1 \leq i \leq n} |k_i| > 4$. By (4.3),

$$\left\| \Box_k \partial_{x_i} \mathscr{A} f \right\|_{L^q_{x_1} L^\infty_{x_2,...,x_n} L^\infty_t (\mathbb{R}^{1+n})} \lesssim \sum_{|\ell_2|, |\ell_i| \leq 1} \left\| \mathscr{F}^{-1}_{\xi_i, \xi_2} \left(\frac{\xi_i}{\xi_2} \eta_{k_i + \ell_i}(\xi_i) \eta_{k_2 + \ell_2}(\xi_2) \right) \right\|_{L^1(\mathbb{R}^2)}$$

$$\times \left\| \Box_k \partial_{x_2} \mathscr{A} f \right\|_{L^q_{x_1} L^\infty_{x_2,...,x_n} L^\infty_t (\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_i \rangle \langle k_2 \rangle^{-1} \langle k_2 \rangle^{1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_2} L^2_{x_1, x_3,...,x_n} L^2_t (\mathbb{R}^{1+n})}$$

$$\lesssim \langle k_2 \rangle^{1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_2} L^2_{x_1, x_3,...,x_n} L^2_t (\mathbb{R}^{1+n})}. \tag{4.23}$$

The other cases $\alpha = 3, ..., n$ is analogous to the case $\alpha = 2$ and we omit the details of the proof. $\square$

Remark 4.6 From the proof of Lemma 4.5, we easily see that

$$\sum_{k \in \mathbb{Z}^n, |k_\alpha| = k_{\max} > 4} \langle k \rangle^\sigma \left\| \Box_k \partial_{x_i} \mathscr{A} f \right\|_{L^q_{x_\beta} L^\infty_{(x_j)_{j \neq \beta}} L^\infty_t (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{k \in \mathbb{Z}^n, |k_\alpha| > 4} \langle k_\alpha \rangle^{\sigma+1/2+1/q} \left\| \Box_k f \right\|_{L^1_{x_\alpha} L^2_{(x_j)_{j \neq \alpha}} L^2_t (\mathbb{R}^{1+n})}. \tag{4.24}$$

5 Proof of Theorem 1.3

Now we briefly indicate the proof of Theorem 1.3. We assume that the nonlinear term takes the form

$$F(u, \nabla u) = \partial_{x_1}(u^{\kappa_1+1}) + \partial_{x_2}(u^{\kappa_2+1}).$$

In order to handle the nonlinear term $\partial_{x_i}(u^{\kappa_i+1})$, we use the space $L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})$ to absorb the derivative ∂_{x_i}. Hence, we introduce the following semi-norms to treat the nonlinearity:

$$\|u\|_{Y_i} = \sum_{k\in\mathbb{Z}^n,\ |k_i|>4} \langle k_i\rangle \|\square_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}, \quad i=1,2.$$

Since (3.12) is a worse estimate in the case $|k_i| \lesssim 1$, we throw away the low frequency part in the ξ_i-direction in the definition of $\|u\|_{Y_i}$. To handle the low frequency part, we use the Strichartz norm:

$$\|u\|_S = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{1/2} \|\square_k u\|_{L^\infty_t L^2_x \cap L^{2+\kappa}_{x,t}(\mathbb{R}^{1+n})}.$$

We emphasize that the Strichartz inequalities (3.24) and (3.30) are better estimates than the smooth effects in (3.22) and (3.25) for the low frequency part, respectively. Using the integral equation

$$u(t) = S(t)u_0 - i\mathscr{A}(\partial_{x_1} u^{\kappa_1+1} + \partial_{x_2} u^{\kappa_2+1}),$$

we have

$$\|u\|_{Y_1} \leq \|S(t)u_0\|_{Y_1} + \|\mathscr{A}(\partial_{x_1} u^{\kappa_1+1})\|_{Y_1} + \|\mathscr{A}(\partial_{x_2} u^{\kappa_2+1})\|_{Y_1}.$$

In view of (3.22), $\|S(t)u_0\|_{Y_1}$ is bounded by $\|u_0\|_{M^{1/2}_{2,1}}$. $\|\mathscr{A}(\partial_{x_1} u^{\kappa_1+1})\|_{Y_1}$ can be handled by using the linear estimates obtained in Section 3. Noticing that

$$u^{\kappa_1+1} = \left(\sum_{k^{(1)},\ldots,k^{(\kappa_1+1)}\in\mathbb{S}^1} + \sum_{k^{(1)},\ldots,k^{(\kappa_1+1)}\in\mathbb{Z}^n\setminus\mathbb{S}^1} \right) \square_{k^{(1)}} u \ldots \square_{k^{(\kappa_1+1)}} u,$$

where $\mathbb{S}^1 = \{k^{(1)},\ldots,k^{(\kappa_1+1)} \in \mathbb{Z}^n : |k_1^{(1)}| \vee \ldots \vee |k_1^{(\kappa_1+1)}| > 4\}$, (3.25) and (3.28) in Corollary 3.11 yield,

$$\|\mathscr{A}(\partial_{x_1} u^{\kappa_1+1})\|_{Y_1} \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}^1} \|\square_k(\square_{k^{(1)}} u \ldots \square_{k^{(\kappa_1+1)}} u)\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t}$$

$$+ \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle^{3/2} \sum_{\mathbb{Z}^n\setminus\mathbb{S}^1} \|\square_k(\square_{k^{(1)}} u \ldots \square_{k^{(\kappa_1+1)}} u)\|_{L^{\frac{2+\kappa}{1+\kappa}}_{x,t}}. \tag{5.1}$$

By performing a nonlinear mapping estimate, we have

$$\|\mathscr{A}(\partial_{x_1} u^{\kappa_1+1})\|_{Y_1} \lesssim \|u\|_{Y_1} \|u\|_{Z_1}^{\kappa_1} + \|u\|_S^{\kappa_1+1}, \tag{5.2}$$

21

where
$$\|u\|_{Z_i} = \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{1/2-1/\kappa} \|\Box_k u\|_{L^\kappa_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})}, \quad i = 1, 2.$$

Unfortunately, $\|\mathscr{A}(\partial_{x_2} u^{\kappa_2+1})\|_{Y_1}$ contains the interaction between the working space $L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})$ and the derivative ∂_{x_2}, which is out of the control of the smooth effect (3.25). So, we look for another way to estimate $\|\mathscr{A}(\partial_{x_2} u^{\kappa_2+1})\|_{Y_1}$. Roughly speaking, our idea is to use the following estimates (see Lemma 4.2):

$$\|\mathscr{F}^{-1}\chi_{\{\xi:\ |\xi_2|\leq|\xi_1|\}}\mathscr{F}\mathscr{A}(\partial_{x_2} f)\|_{Y_1} \lesssim \sum_{k\in\mathbb{Z}^n\ |k_1|>4} \langle k_1 \rangle \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})},$$

$$\|\mathscr{F}^{-1}\chi_{\{\xi:\ |\xi_1|\leq|\xi_2|\}}\mathscr{F}\mathscr{A}(\partial_{x_2} f)\|_{Y_1} \lesssim \sum_{k\in\mathbb{Z}^n\ |k_2|>4} \langle k_2 \rangle \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})},$$

where χ_E denotes the characteristic function on the set E. So, $\|\mathscr{A}(\partial_{x_2} u^{\kappa_2+1})\|_{Y_1}$ has similar bound to $\|\mathscr{A}(\partial_{x_1} u^{\kappa_2+1})\|_{Y_1}$ as in (5.1). Eventually, we have

$$\|\mathscr{A}(\partial_{x_2} u^{\kappa_2+1})\|_{Y_1} \lesssim (\|u\|_{Y_1} + \|u\|_{Y_2})\|u\|_{Z_1}^{\kappa_2} + \|u\|_S^{\kappa_2+1}. \tag{5.3}$$

By using the integral equation, we need to further bound $\|\mathscr{A}\partial_{x_i} u^{\kappa_i+1}\|_{Z_1\cap S}$, $i = 1, 2$. For instance, for the estimate of $\|\mathscr{A}\partial_{x_2} u^{\kappa_2+1}\|_{Z_1}$, we resort to the above idea and consider the following interaction estimate:

$$\|\Box_k \partial_{x_2}\mathscr{A} f\|_{L^\kappa_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_2 \rangle^{1/2} \langle k \rangle^{1/\kappa} \|\Box_k f\|_{L^1_{x_2} L^2_{(x_j)_{j\neq 2}} L^2_t(\mathbb{R}^{1+n})},$$

which leads to that we can bound $\|\mathscr{A}\partial_{x_2} u^{\kappa_2+1}\|_{Z_1}$ by an analogous version of the right-hand side of (5.1), so, by (5.2) (see Lemma 4.5).

Finally, using (3.27) and (3.30), we can get the same estimate of $\|\mathscr{A}\partial_{x_i} u^{\kappa_i+1}\|_S$ as in (5.2) and (5.3), respectively.

Proof of Theorem 1.3. We now give the details of the proof of Theorem 1.3. Denote

$$\rho_1(u) = \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\ |k_i|>4} \langle k_i \rangle \|\Box_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})},$$

$$\rho_2(u) = \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{1/2-1/\kappa} \|\Box_k u\|_{L^\kappa_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})},$$

$$\rho_3(u) = \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k u\|_{L^\infty_t L^2_x \cap L^{2+\kappa}_{x,t}(\mathbb{R}^{1+n})}.$$

Put

$$X := \left\{ u \in \mathscr{S}'(\mathbb{R}^{1+n}) : \ \|u\|_X := \sum_{i=1}^3 \rho_i(u) \leq \delta_0 \right\}.$$

22

We consider the following mapping:

$$\mathscr{T} : u(t) \to S(t)u_0 - \mathrm{i}\mathscr{A}\left(\sum_{i=1}^{n} \lambda_i \partial_{x_i} u^{\kappa_i+1}\right).$$

For convenience, we denote

$$\|u\|_{Y_i} = \sum_{k \in \mathbb{Z}^n, \ |k_i|>4} \langle k_i \rangle \|\Box_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}.$$

In order to estimate $\rho_1(u)$, it suffices to control $\|\cdot\|_{Y_1}$. By (2.8) and Plancherel's identity, we have

$$\|S(t)u_0\|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \|\Box_k D_{x_1}^{-1/2} u_0\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim \sum_{k \in \mathbb{Z}^n} \langle k_1 \rangle^{1/2} \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}.$$

By (3.3), Lemma 3.2, we have

$$\rho_i(S(t)u_0) \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \quad i = 2, 3.$$

Denote

$$\mathbb{S}^{(i)}_{\ell,1} := \{(k^{(1)}, ..., k^{(\kappa_\ell+1)}) \in (\mathbb{Z}^n)^n : \ |k_i^{(1)}| \vee ... \vee |k_i^{(\kappa_\ell+1)}| > 4\},$$
$$\mathbb{S}^{(i)}_{\ell,2} := \{(k^{(1)}, ..., k^{(\kappa_\ell+1)}) \in (\mathbb{Z}^n)^n : \ |k_i^{(1)}| \vee ... \vee |k_i^{(\kappa_\ell+1)}| \le 4\}.$$

Using the frequency-uniform decomposition, we have

$$u^{\kappa_\ell+1} = \sum_{k^{(1)},...,k^{(\kappa_\ell+1)} \in \mathbb{Z}^n} \Box_{k^{(1)}} u ... \Box_{k^{(\kappa_\ell+1)}} u$$

$$= \sum_{\mathbb{S}^{(i)}_{\ell,1}} \Box_{k^{(1)}} u ... \Box_{k^{(\kappa_\ell+1)}} u + \sum_{\mathbb{S}^{(i)}_{\ell,2}} \Box_{k^{(1)}} u ... \Box_{k^{(\kappa_\ell+1)}} u. \qquad (5.4)$$

Using (3.25) and (3.28), we obtain that

$$\|\mathscr{A}\partial_{x_1} u^{\kappa_1+1}\|_{Y_1} \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \sum_{\mathbb{S}^{(1)}_{1,1}} \left\|\Box_k \left(\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_1+1)}} u\right)\right\|_{L^1_{x_1} L^2_{x_2,...,x_n} L^2_t(\mathbb{R}^{1+n})}$$

$$+ \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle^{3/2} \sum_{\mathbb{S}^{(1)}_{1,2}} \left\|\Box_k \left(\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_1+1)}} u\right)\right\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}$$

$$:= I + II. \qquad (5.5)$$

In view of the support property of $\widehat{\square_k u}$, we see that

$$\square_k \left(\square_{k^{(1)}} u ... \square_{k^{(\kappa_1+1)}} u \right) = 0, \quad if \quad |k - k^{(1)} - ... - k^{(\kappa_1+1)}| \geqslant C. \tag{5.6}$$

Hence, by Lemma 3.4,

$$I \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1| > 4} \langle k_1 \rangle \sum_{\mathbb{S}_{1,1}^{(1)}} \left\| \square_{k^{(1)}} u ... \square_{k^{(\kappa_1+1)}} u \right\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2 (\mathbb{R}^{1+n})} \chi_{|k-k^{(1)}-...-k^{(\kappa_1+1)}| \leq C}. \tag{5.7}$$

By Hölder's inequality and $\|\square_k u\|_{L_x^\infty} \lesssim \|\square_k u\|_{L_x^2}$ uniformly holds for all $k \in \mathbb{Z}^n$, we have

$$\left\| \square_{k^{(1)}} u ... \square_{k^{(\kappa_1+1)}} u \right\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2 (\mathbb{R}^{1+n})}$$

$$\leq \left\| \square_{k^{(1)}} u \right\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2 (\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_1+1} \left\| \square_{k^{(i)}} u \right\|_{L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty \cap L_t^\infty L_x^2 (\mathbb{R}^{1+n})}.$$

Since $|k - k^{(1)} - ... - k^{(\kappa_1+1)}| \leq C$ implies that $|k_1 - k_1^{(1)} - ... - k_1^{(\kappa_1+1)}| \leq C$, we see that $|k_1| \leq C \max_{i=1,...,\kappa_1+1} |k_1^{(i)}|$. We may assume that $|k_1^{(1)}| = \max_{i=1,...,\kappa_1+1} |k_1^{(i)}|$ in the summation $\sum_{\mathbb{S}_{1,1}^{(1)}}$ in (5.7) above. So,

$$I \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, \ |k_1^{(1)}| > 4} \langle k_1^{(1)} \rangle \left\| \square_{k^{(1)}} u \right\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2 (\mathbb{R}^{1+n})}$$

$$\times \sum_{k^{(2)},...,k^{(\kappa_1+1)} \in \mathbb{Z}^n} \prod_{i=2}^{\kappa_1+1} \left\| \square_{k^{(i)}} u \right\|_{L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty \cap L_t^\infty L_x^2 (\mathbb{R}^{1+n})}$$

$$\lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_1}. \tag{5.8}$$

In view of (5.6) we easily see that $|k_1| \leq C$ in II of (5.5). Hence,

$$II \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1| > 4} \sum_{\mathbb{S}_{1,2}^{(1)}} \left\| \square_{k^{(1)}} u ... \square_{k^{(\kappa_1+1)}} u \right\|_{L_{x,t}^{(2+\kappa)/(1+\kappa)} (\mathbb{R}^{1+n})} \chi_{|k-k^{(1)}-...-k^{(\kappa_1+1)}| \leq C}$$

$$\lesssim \sum_{\mathbb{S}_{1,2}^{(1)}} \left\| \square_{k^{(1)}} u ... \square_{k^{(\kappa_1+1)}} u \right\|_{L_{x,t}^{(2+\kappa)/(1+\kappa)} (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{\mathbb{S}_{1,2}^{(1)}} \prod_{i=1}^{\kappa_1+1} \left\| \square_{k^{(i)}} u \right\|_{L_{x,t}^{2+\kappa} \cap L_t^\infty L_x^2 (\mathbb{R}^{1+n})} \lesssim \rho_3(u)^{1+\kappa_1}. \tag{5.9}$$

Hence, we have

$$\left\| \mathscr{A} \partial_{x_1} u^{\kappa_1+1} \right\|_{Y_1} \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_1} + \rho_3(u)^{1+\kappa_1}. \tag{5.10}$$

24

Next, we estimate $\|\mathscr{A}\partial_{x_2}u^{\kappa_2+1}\|_{Y_1}$. Let ψ_i be as in Lemma 4.2. For convenience, we write

$$P_i = \mathscr{F}_{\xi_1,\xi_2}^{-1}\psi_i\mathscr{F}_{x_1,x_2}, \quad i = 1, 2. \tag{5.11}$$

We have

$$\|\mathscr{A}\partial_{x_2}u^{\kappa_2+1}\|_{Y_1} \lesssim \|P_1\partial_{x_2}\mathscr{A}u^{\kappa_2+1}\|_{Y_1} + \|P_2\partial_{x_2}\mathscr{A}u^{\kappa_2+1}\|_{Y_1} := III + IV. \tag{5.12}$$

Using the decomposition (5.4),

$$III \le \left\|P_1\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}_{2,1}^{(1)}}(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1}$$

$$+ \left\|P_1\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}_{2,2}^{(1)}}(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1} := III_1 + III_2. \tag{5.13}$$

Applying Lemma 4.2 and then following the same way as in the estimate to (5.7),

$$III_1 \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_1|>4}\langle k_1\rangle\sum_{\mathbb{S}_{2,1}^{(1)}}\|\Box_k\left(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u\right)\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_2}. \tag{5.14}$$

For the estimate of III_2, noticing the fact that $\text{supp}\psi_1 \subset \{\xi : |\xi_2| \le 4|\xi_1|\}$ and using the multiplier estimate, then applying (4.2), we have

$$III_2 \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_1|>4,\ |k_2|\lesssim|k_1|}\langle k_1\rangle^{3/2}\sum_{\mathbb{S}_{2,2}^{(1)}}\|\Box_k\left(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u\right)\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}$$

$$\lesssim \rho_3(u)^{1+\kappa_2}. \tag{5.15}$$

We need to further control IV. Using the decomposition (5.4),

$$IV \le \left\|P_2\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}_{2,1}^{(2)}}(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1}$$

$$+ \left\|P_2\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}_{2,2}^{(2)}}(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1} := IV_1 + IV_2. \tag{5.16}$$

By Lemma 4.2,

$$IV_1 \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_2|>4}\langle k_2\rangle\sum_{\mathbb{S}_{2,1}^{(2)}}\|\Box_k\left(\Box_{k^{(1)}}u...\Box_{k^{(\kappa_2+1)}}u\right)\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{5.17}$$

By symmetry of $k^{(1)}, ..., k^{(\kappa_2+1)}$, we can assume that $|k_2^{(1)}| = \max_{1 \le i \le \kappa_2+1} |k_2^{(i)}|$ in $\mathbb{S}_{2,1}^{(2)}$. Using the same way as in the estimate of I, we have

$$IV_1 \lesssim \sum_{\mathbb{S}_{2,1}^{(2)}, \ |k_2^{(1)}|>4} \langle k_2^{(1)} \rangle \|\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_2+1)}} u\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{5.18}$$

By Hölder's inequality,

$$\|\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_2+1)}} u\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\lesssim \||\Box_{k^{(1)}} u |\Box_{k^{(2)}} u ... \Box_{k^{(\kappa_2+1)}} u|^{1/2}\|_{L_{x,t}^2(\mathbb{R}^{1+n})}$$

$$\times \||\Box_{k^{(2)}} u ... \Box_{k^{(\kappa_2+1)}} u|^{1/2}\|_{L_{x_1}^2 L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}$$

$$\lesssim \|\Box_{k^{(1)}} u\|_{L_{x_2}^\infty L_{x_1,x_3,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_2+1} \|\Box_{k^{(i)}} u\|_{L_{x_2}^{\kappa_2} L_{x_1,x_3,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}$$

$$\times \prod_{i=2}^{\kappa_2+1} \|\Box_{k^{(i)}} u\|_{L_{x_1}^{\kappa_2} L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}. \tag{5.19}$$

In view of the inclusion $L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty \bigcap L_{x,t}^\infty \subset L_{x_1}^{\kappa_2} L_{x_2,...,x_n}^\infty L_t^\infty$, we immediately have

$$IV_1 \lesssim \rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_2}. \tag{5.20}$$

Noticing the fact that $\text{supp}\psi_2 \subset \{\xi : |\xi_2| \ge 2|\xi_1|\}$ and applying (4.2), we have

$$IV_2 \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_2|>4} \langle k_2 \rangle^{3/2} \sum_{\mathbb{S}_{2,2}^{(2)}} \|\Box_k (\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_2+1)}} u)\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{k \in \mathbb{Z}^n, \ |k_2|>4} \sum_{\mathbb{S}_{2,2}^{(2)}} \|\Box_k (\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_2+1)}} u)\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}$$

$$\lesssim \rho_3(u)^{1+\kappa_2}. \tag{5.21}$$

The other terms in $\rho_1(\cdot)$ can be bounded in a similar way. So, we have shown that

$$\rho_1 \left(\mathscr{A}(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}) \right) \lesssim \sum_{i=1}^n \left(\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i} \right). \tag{5.22}$$

We estimate $\rho_2(\cdot)$. Denote

$$\|u\|_{Z_i} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2-1/\kappa} \|\Box_k u\|_{L_{x_i}^\kappa L_{(x_j)_{j \ne i}}^\infty L_t^\infty(\mathbb{R}^{1+n})}. \tag{5.23}$$

We have

$$\rho_2 \left(\mathscr{A}(\sum_{j=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}) \right) \lesssim \sum_{i=1}^n \left\| \mathscr{A}(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}) \right\|_{Z_j}. \tag{5.24}$$

26

Due to the symmetry of $Z_1, ..., Z_n$, it suffices to consider the estimate of $\|\cdot\|_{Z_1}$. Recall that $k_{\max} = |k_1| \vee ... \vee |k_n|$. We have

$$\|v\|_{Z_1} \leq \left(\sum_{k \in \mathbb{Z}^n, \, k_{\max} > 4} + \sum_{k \in \mathbb{Z}^n, \, k_{\max} \leq 4} \right) \langle k \rangle^{1/2 - 1/\kappa} \|\Box_k v\|_{L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty (\mathbb{R}^{1+n})}$$

$$:= \Gamma_1(v) + \Gamma_2(v). \tag{5.25}$$

In view of Lemma 4.4 and Hölder's inequality,

$$\Gamma_2 \left(\mathscr{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i + 1} \right) \right) \leq \sum_{k \in \mathbb{Z}^n, \, k_{\max} \leq 4} \left\| \Box_k \mathscr{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i + 1} \right) \right\|_{L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{i=1}^n \sum_{k^{(1)},...,k^{(\kappa_i+1)} \in \mathbb{Z}^n} \left\| \Box_{k^{(1)}} u ... \Box_{k^{(\kappa_i+1)}} u \right\|_{L_{t,x}^{\frac{2+\kappa_i}{1+\kappa_i}} (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{i=1}^n \sum_{k^{(1)},...,k^{(\kappa_i+1)} \in \mathbb{Z}^n} \left\| \Box_{k^{(1)}} u \right\|_{L_{t,x}^{2+\kappa_i} (\mathbb{R}^{1+n})} ... \left\| \Box_{k^{(\kappa_i+1)}} u \right\|_{L_{t,x}^{2+\kappa_i} (\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{i=1}^n \rho_3(u)^{\kappa_i + 1}. \tag{5.26}$$

It is easy to see that

$$\Gamma_1(v) \leq \left(\sum_{k \in \mathbb{Z}^n, \, |k_1| = k_{\max} > 4} + ... + \sum_{k \in \mathbb{Z}^n, \, |k_n| = k_{\max} > 4} \right) \langle k \rangle^{1/2 - 1/\kappa} \|\Box_k v\|_{L_{x_1}^\kappa L_{x_2,...,x_n}^\infty L_t^\infty (\mathbb{R}^{1+n})}$$

$$:= \Gamma_1^1(v) + ... + \Gamma_1^n(v). \tag{5.27}$$

Using (5.4), Lemmas 4.4 and 4.5, we have

$$\Gamma_1^1 \left(\mathscr{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i + 1} \right) \right)$$

$$\lesssim \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, \, |k_1| > 4} \langle k_1 \rangle \sum_{\mathbb{S}_{i,1}^{(1)}} \left\| \Box_k \left(\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_i+1)}} u \right) \right\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2 (\mathbb{R}^{1+n})}$$

$$+ \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, \, |k_1| > 4} \langle k_1 \rangle^{3/2} \sum_{\mathbb{S}_{i,2}^{(1)}} \left\| \Box_k \left(\Box_{k^{(1)}} u ... \Box_{k^{(\kappa_i+1)}} u \right) \right\|_{L_{t,x}^{(2+\kappa_i)/(1+\kappa_i)} (\mathbb{R}^{1+n})}. \tag{5.28}$$

Using the same way as in (5.8) and (5.9), one easily sees that

$$\Gamma_1^1 \left(\mathscr{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i + 1} \right) \right) \lesssim \sum_{i=1}^n \left(\rho_1(u)(\rho_2(u) + \rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i} \right). \tag{5.29}$$

We estimate $\Gamma_1^2(\cdot)$. By Lemmas 4.4 and 4.5,

$$
\Gamma_1^2\left(\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right)
$$

$$
\lesssim \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\ |k_2|=k_{\max}>4} \langle k\rangle^{1/2-1/\kappa} \|\Box_k\left(\mathscr{A}\partial_{x_i} u^{\kappa_i+1}\right)\|_{L_{x_1}^\kappa L_{x_2,\dots,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}
$$

$$
\lesssim \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\ |k_2|=k_{\max}>4} \langle k_2\rangle \sum_{\mathbb{S}_{i,1}^{(2)}} \|\Box_k\left(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_i+1)}}u\right)\|_{L_{x_2}^1 L_{x_1,x_3,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}
$$

$$
+ \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\ |k_2|>4} \langle k_2\rangle^{3/2} \sum_{\mathbb{S}_{i,2}^{(2)}} \|\Box_k\left(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_i+1)}}u\right)\|_{L_{t,x}^{(2+\kappa_i)/(1+\kappa_i)}(\mathbb{R}^{1+n})}. \qquad (5.30)
$$

This reduces the same estimate as $\Gamma_1^1(\cdot)$. We easily see that $\Gamma_1^i(\cdot)$ for $3 \le i \le n$ can be controlled in a similar way as $\Gamma_1^2(\cdot)$. Hence, we have shown that

$$
\left\|\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right\|_{Z_1} \lesssim \sum_{i=1}^n \left(\rho_1(u)(\rho_2(u)+\rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right). \qquad (5.31)
$$

For the estimates of $\rho_3(\mathscr{A}\partial_{x_i}u^{\kappa_i+1})$, we have from (3.17) and Lemma 3.4 that

$$
\|\Box_k\mathscr{A}\partial_{x_i}f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa}(\mathbb{R}^{1+n})} \lesssim \|\Box_k\partial_{x_i}f\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}
$$

$$
\lesssim \langle k_i\rangle\|\Box_k f\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}. \qquad (5.32)
$$

Hence, using (5.4), (3.30) and (3.27), we obtain that can be controlled by the right hand side of (5.28).

$$
\rho_3(\mathscr{A}\partial_{x_1}u^{\kappa_1+1}) \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_1|\le 4} \langle k_1\rangle^{3/2} \sum_{k^{(1)},\dots,k^{(\kappa_1+1)}\in\mathbb{Z}^n} \|\Box_k\left(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\right)\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}
$$

$$
+ \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_{1,1}^{(1)}} \|\Box_k\left(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\right)\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}
$$

$$
+ \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}_{1,2}^{(1)}} \|\Box_k\left(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\right)\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}(\mathbb{R}^{1+n})}.
$$

$$
(5.33)
$$

By (5.8) and (5.9), we have

$$
\rho_3(\mathscr{A}\partial_{x_1}u^{\kappa_1+1}) \lesssim \sum_{i=1}^n \left(\rho_1(u)(\rho_2(u)+\rho_3(u))^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right). \qquad (5.34)
$$

28

Hence, we have shown that

$$\|\mathscr{T}u\|_X \lesssim \|u_0\|_{M_{2,1}^{1/2}} + \sum_{i=1}^{n} \|u\|_X^{1+\kappa_i}. \tag{5.35}$$

Using a standard contraction mapping argument, we can finish the proof of Theorem 1.3. □

6 Proof of Theorem 1.1

Roughly speaking, we will prove our Theorem 1.1 by following some ideas as in the proof of Theorem 1.3. However, due to the nonlinearity contains $u^{\kappa+1}$, and $(\nabla u)^\nu$ and $u^\kappa (\nabla u)^\nu$ as special cases, the proof of Theorem 1.3 can not be directly applied. We construct the space X as follows. Denote

$$\varrho_1^{(i)}(u) = \sum_{k \in \mathbb{Z}^n, \ |k_i| > 4} \langle k_i \rangle \|\Box_k u\|_{L_{x_i}^\infty L_{(x_j)_{j \neq i}}^2 L_t^2(\mathbb{R}^{1+n})},$$

$$\varrho_2^{(i)}(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2 - 1/m} \|\Box_k u\|_{L_{x_i}^m L_{(x_j)_{j \neq i}}^\infty L_t^\infty(\mathbb{R}^{1+n})},$$

$$\varrho_3^{(i)}(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k u\|_{L_{x,t}^{2+m} \cap L_t^\infty L_x^2(\mathbb{R}^{1+n})}.$$

Put

$$X := \left\{ u \in \mathscr{S}'(\mathbb{R}^{1+n}) : \ \|u\|_X := \sum_{\ell=1}^{3} \sum_{\alpha=0,1} \sum_{i,j=1}^{n} \varrho_\ell^{(i)}(\partial_{x_j}^\alpha u) \leq \delta \right\}.$$

Considering the following mapping:

$$\mathscr{T} : u(t) \to S(t)u_0 - i\mathscr{A} F(u, \bar{u}, \nabla u, \nabla \bar{u}),$$

we will show that $\mathscr{T} : X \to X$ is a contraction mapping.

Since $\|u\|_X = \|\bar{u}\|_X$, we may assume, without loss of generality that

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = F(u, \nabla u) := \sum_{m+1 \leq \kappa + |\nu| < \infty} c_{\kappa\nu} u^\kappa (\nabla u)^\nu,$$

where $(\nabla u)^\nu = u_{x_1}^{\nu_1} ... u_{x_n}^{\nu_n}$. For the sake of convenience, we denote

$$v_1 = ... = v_\kappa = u, \quad v_{\kappa+1} = ... = v_{\kappa+\nu_1} = u_{x_1}, ..., v_{\kappa+|\nu|-\nu_n+1} = ... = v_{\kappa+|\nu|} = u_{x_n}.$$

By (2.8), for $\alpha = 0, 1$,

$$\varrho_1^{(i)}(\partial_{x_j}^\alpha S(t) u_0) \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_i| > 4} \langle k_i \rangle^{1/2} \langle k_j \rangle \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \le \|u_0\|_{M_{2,1}^{3/2}}.$$

By (3.23), (3.24), we have for $\alpha = 0, 1$,

$$\varrho_2^{(i)}(\partial_{x_j}^\alpha S(t) u_0) + \varrho_3^{(i)}(\partial_{x_j}^\alpha S(t) u_0) \lesssim \|u_0\|_{M_{2,1}^{3/2}}.$$

Hence,

$$\|S(t) u_0\|_X \lesssim \|u_0\|_{M_{2,1}^{3/2}}.$$

In order to estimate $\varrho_1^{(i)}(\mathscr{A} \partial_{x_j}^\alpha (v_1 ... v_{\kappa + |\nu|}))$, $i, j = 1, ..., n$, it suffices to estimate $\varrho_1^{(1)}(\mathscr{A} \partial_{x_1}^\alpha (v_1 ... v_{\kappa + |\nu|}))$ and $\varrho_1^{(1)}(\mathscr{A} \partial_{x_2}^\alpha (v_1 ... v_{\kappa + |\nu|}))$. Similarly as in (5.4), we will use the decomposition

$$\Box_k(v_1 ... v_{\kappa + |\nu|}) = \sum_{\mathbb{S}_1^{(i)}} \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa + |\nu|)}} v_{\kappa + |\nu|} \right)$$
$$+ \sum_{\mathbb{S}_2^{(i)}} \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa + |\nu|)}} v_{\kappa + |\nu|} \right), \tag{6.1}$$

where

$$\mathbb{S}_1^{(i)} := \{ (k^{(1)}, ..., k^{(\kappa + |\nu|)}) : \ |k_i^{(1)}| \vee ... \vee |k_i^{(\kappa + |\nu|)}| > 4 \},$$
$$\mathbb{S}_2^{(i)} := \{ (k^{(1)}, ..., k^{(\kappa + |\nu|)}) : \ |k_i^{(1)}| \vee ... \vee |k_i^{(\kappa + |\nu|)}| \le 4 \}.$$

In view of (3.12) and (3.19),

$$\varrho_1^{(1)}(\mathscr{A} \partial_{x_1}^\alpha (v_1 ... v_{\kappa + |\nu|}))$$
$$\lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1| > 4} \langle k_1 \rangle \sum_{\mathbb{S}_1^{(1)}} \left\| \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa + |\nu|)}} v_{\kappa + |\nu|} \right) \right\|_{L_{x_1}^1 L_{x_2, ..., x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$
$$+ \sum_{k \in \mathbb{Z}^n, \ |k_1| > 4} \langle k_1 \rangle^{3/2} \sum_{\mathbb{S}_2^{(1)}} \left\| \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa + |\nu|)}} v_{\kappa + |\nu|} \right) \right\|_{L_{t,x}^{\frac{\kappa + |\nu| + 1}{\kappa + |\nu|}}(\mathbb{R}^{1+n})}$$
$$:= I + II. \tag{6.2}$$

Similar to (5.8),

$$I \lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, \ |k_1^{(1)}| > 2} \langle k_1^{(1)} \rangle \|\Box_{k^{(1)}} v_1\|_{L_{x_1}^\infty L_{x_2, ..., x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$\times \sum_{k^{(2)},...,k^{(\kappa+|\nu|)}\in\mathbb{Z}^n} \prod_{i=2}^{\kappa+|\nu|} \left\|\square_{k^{(i)}}v_i\right\|_{L_{x_1}^{\kappa+|\nu|-1}L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}. \tag{6.3}$$

By Hölder's inequality and Lemma 3.3,

$$\left\|\square_{k^{(i)}}v_i\right\|_{L_{x_1}^{\kappa+|\nu|-1}L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}$$
$$\leq \left\|\square_{k^{(i)}}v_i\right\|_{L_{x_1}^m L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}^{\frac{m}{\kappa+|\nu|-1}} \left\|\square_{k^{(i)}}v_i\right\|_{L_{x,t}^\infty(\mathbb{R}^{1+n})}^{1-\frac{m}{\kappa+|\nu|-1}}$$
$$\lesssim \left\|\square_{k^{(i)}}v_i\right\|_{L_{x_1}^m L_{x_2,...,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}^{\frac{m}{\kappa+|\nu|-1}} \left\|\square_{k^{(i)}}v_i\right\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})}^{1-\frac{m}{\kappa+|\nu|-1}}. \tag{6.4}$$

Hence, noticing that $v_i = u$ or $v_i = u_{x_j}$, we have from (6.3) and (6.4),

$$I \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.5}$$

Similar to (5.9), we see that $|k_1| \leq C$ in the summation of II. Again, in view of Hölder's inequality and Lemma (3.3),

$$\left\|\square_{k^{(1)}}v_1...\square_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|}\right\|_{L_{x,t}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})} \leq \prod_{i=1}^{\kappa+|\nu|} \left\|\square_{k^{(i)}}v_i\right\|_{L_{x,t}^{\kappa+|\nu|+1}(\mathbb{R}^{1+n})}$$
$$\lesssim \prod_{i=1}^{\kappa+|\nu|} \left\|\square_{k^{(i)}}v_i\right\|_{L_{x,t}^{2+m} \cap L_t^\infty L_x^2(\mathbb{R}^{1+n})}. \tag{6.6}$$

Hence, using a similar way as in (5.9),

$$II \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.7}$$

We now give the estimate of $\varrho_1^{(1)}(\mathscr{A}\partial_{x_2}^\alpha(v_1...v_{\kappa+|\nu|}))$. Since we have obtained the estimate in the case $\alpha = 0$, it suffices to consider the case $\alpha = 1$. Let ψ_i $(i = 1,2)$ be as in Lemma 4.2 and $P_i = \mathscr{F}^{-1}\psi_i\mathscr{F}$. We have

$$\varrho_1^{(1)}(\mathscr{A}\partial_{x_2}(v_1...v_{\kappa+|\nu|}))$$
$$\leq \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \left\|P_1\square_k(\mathscr{A}\partial_{x_2}(v_1...v_{\kappa+|\nu|}))\right\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$
$$+ \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \left\|P_2\square_k(\mathscr{A}\partial_{x_2}(v_1...v_{\kappa+|\nu|}))\right\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$
$$:= III + IV. \tag{6.8}$$

Using the decomposition (6.1),

$$III \leq \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_1^{(1)}} \left\|P_1\square_k(\mathscr{A}\partial_{x_2}(\square_{k^{(1)}}v_1...\square_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|}))\right\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$

31

$$+ \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \sum_{\mathbb{S}_2^{(1)}} \|P_1 \square_k (\mathscr{A} \partial_{x_2} (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}))\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$

$$:= III_1 + III_2. \tag{6.9}$$

By Lemma 4.2,

$$III_1 \lesssim \sum_{\mathbb{S}_1^{(1)}} \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \|\square_k (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2}. \tag{6.10}$$

By symmetry, we may assume $|k_1^{(1)}| = \max(|k_1^{(1)}|, ..., |k_1^{(\kappa+|\nu|)}|)$ in $\mathbb{S}_1^{(1)}$. Hence,

$$III_1 \lesssim \sum_{\mathbb{S}_1^{(1)}, \ |k_1^{(1)}|>4} \langle k_1^{(1)} \rangle \|\square_{k^{(1)}} v_1\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2} \prod_{i=2}^{\kappa+|\nu|} \|\square_{k^{(i)}} v_i\|_{L_{x_1}^{\kappa+|\nu|-1} L_{x_2,...,x_n}^\infty L_t^\infty}$$

$$\lesssim \varrho_1^{(1)}(v_1) \prod_{i=2}^{\kappa+|\nu|} (\varrho_2^{(1)}(v_i) + \varrho_3^{(1)}(v_i)) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.11}$$

Applying (4.2) and using a similar way as in (5.15),

$$III_2 \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1|>4, \ |k_2|\lesssim|k_1|} \langle k_1 \rangle^{3/2} \sum_{\mathbb{S}_2^{(1)}} \|\square_k (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\|_{L_{t,x}^{(2+m)/(1+m)}(\mathbb{R}^{1+n})}$$

$$\lesssim \prod_{i=1}^{\kappa+|\nu|} \varrho_3^{(1)}(v_i) \leq \|u\|_X^{\kappa+|\nu|}. \tag{6.12}$$

So, we have shown that

$$III \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.13}$$

Now we estimate IV. Using the decomposition (6.1),

$$IV \leq \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \sum_{\mathbb{S}_1^{(2)}} \|P_2 \square_k (\mathscr{A} \partial_{x_2} (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}))\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$

$$+ \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \sum_{\mathbb{S}_2^{(2)}} \|P_2 \square_k (\mathscr{A} \partial_{x_2} (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}))\|_{L_{x_1}^\infty L_{x_2,...,x_n}^2 L_t^2}$$

$$:= IV_1 + IV_2. \tag{6.14}$$

By Lemma 4.2,

$$IV_1 \lesssim \sum_{\mathbb{S}_1^{(2)}} \sum_{k \in \mathbb{Z}^n, \ |k_2|>4} \langle k_2 \rangle \|\square_k (\square_{k^{(1)}} v_1 ... \square_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2}. \tag{6.15}$$

In view of the symmetry, one can bound IV_1 by using the same way as that of III_1 and as in (5.17)–(5.20):

$$IV_1 \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.16}$$

For the estimate of IV_2, we apply (4.2),

$$IV_2 \lesssim \sum_{k\in\mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle^{1/2}\langle k_2\rangle \sum_{\mathbb{S}_2^{(2)}} \|P_2\Box_k(\Box_{k^{(1)}}v_1...\Box_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|})\|_{L_{t,x}^{(2+m)/(1+m)}(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{\mathbb{S}_2^{(2)}} \|\Box_{k^{(1)}}v_1...\Box_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|}\|_{L_{t,x}^{(2+m)/(1+m)}(\mathbb{R}^{1+n})} \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.17}$$

Hence, in view of (6.16) and (6.17), we have

$$IV \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.18}$$

Collecting (6.5), (6.7), (6.13), (6.18), we have shown that

$$\sum_{\alpha=0,1}\sum_{i,j=1}^n \varrho_1^{(i)}(\mathscr{A}\partial_{x_j}^\alpha(u^\kappa(\nabla u)^\nu)) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.19}$$

Lemma 6.1 *Let $s \geqslant 0$, $1 \le p, p_i, \gamma, \gamma_i \le \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + ... + \frac{1}{\gamma_N}. \tag{6.20}$$

Then

$$\sum_{k\in\mathbb{Z}^n}\langle k_1\rangle^s \|\Box_k(u_1...u_N)\|_{L_t^\gamma L_x^p(\mathbb{R}^{1+n})} \lesssim \prod_{i=1}^N \left(\sum_{k\in\mathbb{Z}^n}\langle k_1\rangle^s\|\Box_k u_i\|_{L_t^{\gamma_i} L_x^{p_i}(\mathbb{R}^{1+n})}\right). \tag{6.21}$$

Proof. See [30], Lemma 7.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we consider the estimates of $\varrho_2^{(1)}(\mathscr{A}(u^\kappa(\nabla u)^\nu))$ and $\varrho_3^{(1)}(\mathscr{A}(u^\kappa(\nabla u)^\nu))$. In view of (3.30) and (3.20),

$$\sum_{j=2,3} \varrho_j^{(1)}(\mathscr{A}(u^\kappa(\nabla u)^\nu)) \lesssim \sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k(u^\kappa(\nabla u)^\nu)\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}. \tag{6.22}$$

We use Lemma 6.1 to control the right hand side of (6.22):

$$\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k(v_1...v_{\kappa+|\nu|})\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}$$

33

$$\lesssim \prod_{i=1}^{m+1} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k v_i\|_{L_{t,x}^{2+m}(\mathbb{R}^{1+n})} \right) \prod_{i=m+2}^{\kappa+|\nu|} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k v_i\|_{L_{t,x}^{\infty}(\mathbb{R}^{1+n})} \right)$$

$$\lesssim \prod_{i=1}^{m+1} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k v_i\|_{L_{t,x}^{2+m}(\mathbb{R}^{1+n})} \right) \prod_{i=m+2}^{\kappa+|\nu|} \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/2} \|\Box_k v_i\|_{L_t^{\infty} L_x^2(\mathbb{R}^{1+n})} \right)$$

$$\lesssim \prod_{i=1}^{\kappa+|\nu|} \varrho_3^{(1)}(v_i) \leq \|u\|_X^{\kappa+|\nu|}. \tag{6.23}$$

We estimate $\varrho_2^{(1)}(\mathscr{A} \partial_{x_1}(u^\kappa (\nabla u)^\nu))$. Recall that $k_{\max} = |k_1| \vee ... \vee |k_n|$.

$$\varrho_2^{(1)}(\mathscr{A} \partial_{x_1}(v_1 ... v_{\kappa+|\nu|}))$$

$$\lesssim \sum_{k \in \mathbb{Z}^n, \ k_{\max}>4} \langle k \rangle^{1/2-1/m} \|\Box_k \mathscr{A} \partial_{x_1} \left(v_1 ... v_{\kappa+|\nu|} \right) \|_{L_{x_1}^m L_{x_2,...,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})}$$

$$+ \sum_{k \in \mathbb{Z}^n, \ k_{\max}\leq 4} \langle k \rangle^{1/2-1/m} \|\Box_k \mathscr{A} \partial_{x_1} \left(v_1 ... v_{\kappa+|\nu|} \right) \|_{L_{x_1}^m L_{x_2,...,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})}$$

$$:= V + VI. \tag{6.24}$$

By (3.20) and Lemma 6.1, we have

$$VI \lesssim \sum_{k \in \mathbb{Z}^n} \|\Box_k (v_1 ... v_{\kappa+|\nu|})\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})} \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.25}$$

It is easy to see that

$$V \lesssim \left(\sum_{k \in \mathbb{Z}^n, \ |k_1|=k_{\max}>4} + ... + \sum_{k \in \mathbb{Z}^n, \ |k_n|=k_{\max}>4} \right) \langle k \rangle^{1/2-1/m}$$

$$\times \|\Box_k \mathscr{A} \partial_{x_1} \left(v_1 ... v_{\kappa+|\nu|} \right) \|_{L_{x_1}^m L_{x_2,...,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} := \Upsilon_1(u) + ... + \Upsilon_n(u). \tag{6.26}$$

Applying the decomposition (6.1) and Lemmas 4.4 and 4.5, we obtain that

$$\Upsilon_1(u) \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle \sum_{\mathbb{S}_1^{(1)}} \|\Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|} \right) \|_{L_{x_1}^1 L_{x_2,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$

$$+ \sum_{k \in \mathbb{Z}^n, \ |k_1|>4} \langle k_1 \rangle^{3/2} \sum_{\mathbb{S}_2^{(1)}} \|\Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|} \right) \|_{L_{t,x}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})}, \tag{6.27}$$

which reduces to the case $\alpha = 1$ in (6.2). So,

$$\Upsilon_1(u) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.28}$$

34

Again, in view of Lemmas 4.4 and 4.5,

$$\Upsilon_2(u) \lesssim \sum_{k \in \mathbb{Z}^n, \ |k_2|>4} \langle k_2 \rangle \sum_{\mathbb{S}_1^{(2)}} \left\| \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|} \right) \right\|_{L_{x_2}^1 L_{x_1,x_3,...,x_n}^2 L_t^2(\mathbb{R}^{1+n})}$$
$$+ \sum_{k \in \mathbb{Z}^n, \ |k_2|>4} \langle k_2 \rangle^{3/2} \sum_{\mathbb{S}_2^{(2)}} \left\| \Box_k \left(\Box_{k^{(1)}} v_1 ... \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|} \right) \right\|_{L_{t,x}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})}, \quad (6.29)$$

which reduces to the same estimate as $\Upsilon_1(u)$. Using the same way as $\Upsilon_2(u)$, we can get the estimates of $\Upsilon_3(u), ..., \Upsilon_n(u)$. So,

$$\varrho_2^{(1)}(\mathscr{A}\partial_{x_1}(v_1...v_{\kappa+|\nu|})) \lesssim \|u\|_X^{\kappa+|\nu|}. \quad (6.30)$$

We need to further bound $\varrho_2^{(1)}(\mathscr{A}\partial_{x_i}(v_1...v_{\kappa+|\nu|}))$, $i = 2, ..., n$, which is essentially the same as $\varrho_2^{(1)}(\mathscr{A}\partial_{x_1}(v_1...v_{\kappa+|\nu|}))$. Indeed, it is easy to see that (6.24) holds if we substitute ∂_{x_1} with ∂_{x_i}. Moreover, using Lemmas 6.1, 4.4 and 4.5, we easily get that

$$\varrho_2^{(1)} \left(\mathscr{A}(\partial_{x_i}(v_1...v_{\kappa+|\nu|})) \right) \lesssim \|u\|_X^{\kappa+|\nu|}. \quad (6.31)$$

By Lemma 3.4, (3.17), we see that

$$\|\Box_k \mathscr{A}\partial_{x_1} f\|_{L_t^\infty L^2 \cap L_{t,x}^{2+m}(\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle \|\Box_k f\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}. \quad (6.32)$$

Hence, in view of (3.28) and (3.18), repeating the procedure as in the estimates of $\rho_3(u)$ in Theorem 1.3, $\varrho_3^{(1)}(\mathscr{A}\partial_{x_1}(v_1...v_{\kappa+|\nu|}))$ can be controlled by the right hand side of (6.27) and (6.25). Summarizing the estimates as in the above, we have shown that[2]

$$\|\mathscr{T}u\|_X \le C\|u_0\|_{M^{3/2}} + \sum_{m+1 \le \ell < \infty} \ell^{2n+2} C^\ell \|u\|_X^\ell. \quad (6.33)$$

Applying a standard contraction mapping argument, we can prove our result.

7 Proofs of Theorems 1.4 and 1.2

Proof of Theorem 1.4. For convenience, we denote

$$\rho_1(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n, \ |k_i|>4} \langle k_i \rangle^2 \|\Box_k u\|_{L_{x_i}^\infty L_{(x_j)_{j \ne i}}^2 L_t^2(\mathbb{R}^{1+n})}$$

---

[2]Notice that $|c_\beta| \le C^{|\beta|}$.

$$\rho_2(u) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}^n} \|\Box_k u\|_{L_{x_i}^{\kappa} L_{(x_j)_{j \neq i}}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})}$$

$$\rho_3(u) = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\Box_k u\|_{L_t^{\infty} L_x^2 \cap L_t^3 L_x^6(\mathbb{R}^{1+n})}.$$

Comparing the definitions of $\rho_i(u)$ with those of Section 5, we see that here we drop the regularity $\langle k_i \rangle^{1/2 - 1/\kappa}$ in $\rho_2(u)$ and we add 1-order regularity in $\rho_1(u)$ and $\rho_3(u)$. The estimates for $\rho_1(\mathscr{T}u)$ and $\rho_3(\mathscr{T}u)$ can be shown by following the same way as in the Section 5 (It is worth to notice that in Section 5, when we estimate $\rho_1(\mathscr{T}u)$ and $\rho_3(\mathscr{T}u)$, we can replace $\rho_2(u)$ defined here to substitute that in Section 5). We also need to point out that for $n \geqslant 2$, $2/3 < n(1/2 - 1/6)$ and so, $\|\cdot\|_{L_t^3 L_x^6(\mathbb{R}^{1+n})}$ is a Strichartz norm. Moreover,

$$\|\Box_k u\|_{L_{x,t}^{2+p}} \lesssim \|\Box_k u\|_{L_t^{\infty} L_x^2 \cap L_t^3 L_x^6(\mathbb{R}^{1+n})}$$

uniformly holds for all $k \in \mathbb{Z}^n$ and $2 \leq p \leq \infty$.

Noticing that in the proof of Theorem 1.3, we do not know if the following two inequalities hold for $m = 2$,

$$\|\Box_k \mathscr{A} \partial_{x_1} f\|_{L_{x_1}^m L_{x_2,\ldots,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle^{1/2 + 1/m} \|\Box_k f\|_{L_{x_1}^1 L_{x_2,\ldots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}, \tag{7.1}$$

$$\|\Box_k \partial_{x_i} \mathscr{A} f\|_{L_{x_1}^m L_{x_2,\ldots,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} \lesssim \langle k_i \rangle^{1/2} \langle k_1 \rangle^{1/m} \|\Box_k f\|_{L_{x_i}^1 L_{(x_j)_{j \neq i}}^2 L_t^2(\mathbb{R}^{1+n})}. \tag{7.2}$$

So, in the case $m = 2$, we need to find another way to estimate $\rho_2(\mathscr{T}u)$. Our solution is to apply the following estimate as in (3.20):

$$\|\Box_k \mathscr{A} f\|_{L_{x_1}^2 L_{x_2,\ldots,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} \lesssim \langle k_1 \rangle^{1/2} \|\Box_k f\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})}. \tag{7.3}$$

It follows that for any $\kappa \geqslant 2$,

$$\sum_{k \in \mathbb{Z}^n} \|\Box_k \mathscr{A} \partial_{x_i} u^{\kappa+1}\|_{L_{x_1}^2 L_{x_2,\ldots,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\Box_k u^{\kappa+1}\|_{L_t^1 L_x^2(\mathbb{R}^{1+n})}. \tag{7.4}$$

Using Lemma 6.1, one has that

$$\sum_{k \in \mathbb{Z}^n} \|\Box_k \mathscr{A} \partial_{x_i} u^{\kappa+1}\|_{L_{x_1}^2 L_{x_2,\ldots,x_n}^{\infty} L_t^{\infty}(\mathbb{R}^{1+n})} \lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\Box_k u\|_{L_t^{\kappa+1} L_x^{2(\kappa+1)}(\mathbb{R}^{1+n})} \right)^{\kappa+1}$$

$$\lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{3/2} \|\Box_k u\|_{L_t^3 L_x^6 \cap L_{x,t}^{\infty}(\mathbb{R}^{1+n})} \right)^{\kappa+1}$$

$$\lesssim \rho_3(u)^{1+\kappa}. \tag{7.5}$$

Using (7.5), the estimates of $\rho_2(\mathscr{T}u)$ is also obtained. $\qquad\square$

Proof of Theorem 1.2. We can follow the proof of Theorems 1.4 and 1.1 to get the proof and we omit the details of the proof. $\qquad\square$

A   Appendix

In this section, we generalize the Christ-Kiselev Lemma [6] to anisotropic Lebesgue spaces. Our idea follows Molinet and Ribaud [20], and Smith and Sogge [24]. Denote

$$Tf(t) = \int_{-\infty}^{\infty} K(t,t')f(t')dt', \quad T_{re}f(t) = \int_0^t K(t,t')f(t')dt'. \qquad (A.1)$$

If $T: Y_1 \to X_1$ implies that $T_{re}: Y_1 \to X_1$, then $T: Y_1 \to X_1$ is said to be a well restriction operator.

Proposition A.1 *Let T be as in (A.1). We have the following results.*

(1) *If $\wedge_{i=1}^3 p_i > (\vee_{i=1}^3 q_i) \vee (q_1 q_3/q_2)$, then $T: L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}^3) \to L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)$ is a well restriction operator.*

(2) *If $p_1 > (\vee_{i=1}^3 q_i) \vee (q_1 q_3/q_2)$, then $T: L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}^3) \to L_t^{p_1} L_{x_1}^{p_2} L_{x_2}^{p_3}(\mathbb{R}^3)$ is a well restriction operator.*

(3) *If $q_1 < \wedge_{i=1}^3 p_i$, then $T: L_t^{q_1} L_{x_1}^{q_2} L_{x_2}^{q_3}(\mathbb{R}^3) \to L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)$ is a well restriction operator.*

(4) *If $\wedge_{i=1}^3 p_i > (\vee_{i=1}^3 q_i) \vee (q_1 q_3/q_2)$, then $T: L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}^3) \to L_{x_2}^{p_2} L_{x_1}^{p_1} L_t^{p_3}(\mathbb{R}^3)$ is a well restriction operator.*

Let $f \in L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}^3)$ so that $\|f\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}^3)} = 1$. Define $F: \mathbb{R} \to [0,1]$ by

$$F(t) := \left\| \left(\int_{-\infty}^t |f(s,x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2}}^{q_1} \qquad (A.2)$$

Lemma A.2 *Let $I \subset [0,1]$ is an interval, then it holds:*

$$\|\chi_{F^{-1}(I)} f\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R}\times\mathbb{R}^2)} \lesssim |I|^{\frac{q_2}{q_1 q_3} \wedge \frac{1}{q_1} \wedge \frac{1}{q_2} \wedge \frac{1}{q_3}} \qquad (A.3)$$

37

Proof. For any $I = (A, B) \subset [0, 1]$, there exist $t_1, t_2 \in \mathbb{R}$ satisfying

$$A = \left\| \left(\int_{-\infty}^{t_1} |f(s, x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2}}, \quad B = \left\| \left(\int_{-\infty}^{t_2} |f(s, x)|^{q_3} ds \right)^{1/q_3} \right\|_{L_{x_1}^{q_1} L_{x_2}^{q_2}}$$

and $F^{-1}(I) = (t_1, t_2)$. For $x = (x_1, x_2)$, we define $J(t, x)$ and $E(t, x_1)$ by:

$$J(t, x) = \left(\int_{-\infty}^{t} |f(s, x)|^{q_3} ds \right)^{1/q_3}, \quad E(t, x_1) = \left(\int J(t, x)^{q_2} dx_2 \right)^{1/q_2}. \tag{A.4}$$

It is well known that for $a \geqslant b > 0$,

$$r^a - s^a \leqslant C(r^b - s^b)(r^{a-b} + s^{a-b}), \quad 0 \leqslant s \leqslant r, \tag{A.5}$$

and for $0 < a \leq b$,

$$r^a - s^a \leqslant (r^b - s^b)^{a/b}, \quad 0 \leqslant s \leqslant r. \tag{A.6}$$

We divide the proof into the following four cases.

Case 1. $q_3 \geqslant q_2 \geqslant q_1$. From (A.5) we have

$$\|\chi_{F^{-1}(I)} f(\cdot, x)\|_{L_t^{q_3}}^{q_3} \lesssim (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) J(\infty, x)^{q_3 - q_2} \tag{A.7}$$

Recalling the assumption $\|f\|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R} \times \mathbb{R}^2)} = 1$, by (A.7), (A.4), (A.5) and Hölder inequality, we have

$$\int \left(\int \|\chi_{F^{-1}(I)} f(\cdot, x)\|_{L_t^{q_3}}^{q_2} dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

$$\lesssim \int \left(\int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{\frac{q_2}{q_3}} J(\infty, x)^{(q_3 - q_2)\frac{q_2}{q_3}} dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

$$\leqslant \int \left(\left\| (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{\frac{q_2}{q_3}} \right\|_{L_{x_2}^{\frac{q_3}{q_2}}} \left\| J(\infty, x)^{(q_3 - q_2)\frac{q_2}{q_3}} \right\|_{L_{x_2}^{1/(1 - q_2/q_3)}} \right)^{\frac{q_1}{q_2}} dx_1$$

$$= \int \left(E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_1}{q_3}} \left(E(\infty, x_1) \right)^{\frac{(q_3 - q_2)q_1}{q_3}} dx_1 \tag{A.8}$$

$$\lesssim \int \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_1}{q_3}} \left(E(\infty, x_1) \right)^{\frac{(q_2 - q_1)q_1}{q_3}} \left(E(\infty, x_1) \right)^{\frac{(q_3 - q_2)q_1}{q_3}} dx_1$$

$$\leqslant \left\| \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_1}{q_3}} \right\|_{L_{x_1}^{q_3/q_1}} \left\| E(\infty, x_1)^{\frac{(q_3 - q_1)q_1}{q_3}} \right\|_{L_{x_1}^{1/(1 - q_1/q_3)}} \tag{A.9}$$

$$\leqslant (F(t_2) - F(t_1))^{\frac{q_1}{q_3}} F(\infty)^{1 - q_1/q_3} \leqslant |I|^{\frac{q_1}{q_3}}. \tag{A.10}$$

Case 2. $q_3 \geqslant q_2, q_2 < q_1$. From (A.8) and (A.6), we have

$$\int \left(\int \|\chi_{F^{-1}(I)} f(\cdot, x)\|_{L_t^{q_3}}^{q_2} dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

38

$$\lesssim \int \left(E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_1}{q_3}} \left(E(\infty, x_1) \right)^{\frac{(q_3 - q_2)q_1}{q_3}} dx_1$$

$$\leqslant \int \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_2}{q_3}} \left(E(\infty, x_1) \right)^{\frac{(q_3 - q_2)q_1}{q_3}} dx_1$$

$$\leqslant \left\| \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_2}{q_3}} \right\|_{L_{x_1}^{q_3/q_2}} \left\| E(\infty, x_1)^{\frac{(q_3 - q_2)q_1}{q_3}} \right\|_{L_{x_1}^{1/(1 - q_2/q_3)}}$$

$$\leqslant (F(t_2) - F(t_1))^{\frac{q_2}{q_3}} F(\infty)^{1 - q_2/q_3} \leqslant |I|^{\frac{q_2}{q_3}}. \tag{A.11}$$

Case 3. $q_3 < q_2 \leqslant q_1$. From (A.6), we have

$$\| \chi_{F^{-1}(I)} f(\cdot, x) \|_{L_t^{q_3}}^{q_3} \leqslant (J(t_2, x)^{q_2} - J(t_1, x)^{q_2})^{q_3/q_2} \tag{A.12}$$

Using (A.6) again, we have

$$\int \left(\int \| \chi_{F^{-1}(I)} f(\cdot, x) \|_{L_t^{q_3}}^{q_2} dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

$$\leqslant \int \left(\int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) dx_2 \right)^{\frac{q_1}{q_2}} dx_1 \tag{A.13}$$

$$= \int \left(E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_1}{q_2}} dx_1$$

$$\leq \int \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right) dx_1$$

$$= F(t_2) - F(t_1) = |I|. \tag{A.14}$$

Case 4. $q_3 < q_2, q_2 > q_1$. From (A.13), (A.5) and Hölder inequality we have

$$\int \left(\int \| \chi_{F^{-1}(I)} f(\cdot, x) \|_{L_t^{q_3}}^{q_2} dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

$$\leqslant \int \left(\int (J(t_2, x)^{q_2} - J(t_1, x)^{q_2}) dx_2 \right)^{\frac{q_1}{q_2}} dx_1$$

$$= \int \left(E(t_2, x_1)^{q_2} - E(t_1, x_1)^{q_2} \right)^{\frac{q_1}{q_2}} dx_1$$

$$\leqslant \int \left(E(t_2, x_1)^{q_1} - E(t_1, x_1)^{q_1} \right)^{\frac{q_1}{q_2}} E(\infty, x_1)^{\frac{q_1(q_2 - q_1)}{q_2}} dx_1$$

$$\leqslant \left(F(t_2) - F(t_1) \right)^{\frac{q_1}{q_2}} = |I|^{\frac{q_1}{q_2}}. \tag{A.15}$$

From (A.10), (A.11), (A.14) and (A.15) we get

$$\| \chi_{F^{-1}(I)} f \|_{L_{x_1}^{q_1} L_{x_2}^{q_2} L_t^{q_3}(\mathbb{R} \times \mathbb{R}^2)} \leqslant C |I|^{\frac{1}{q_1} \wedge \frac{1}{q_2} \wedge \frac{1}{q_3} \wedge \frac{q_2}{q_1 q_3}}, \tag{A.16}$$
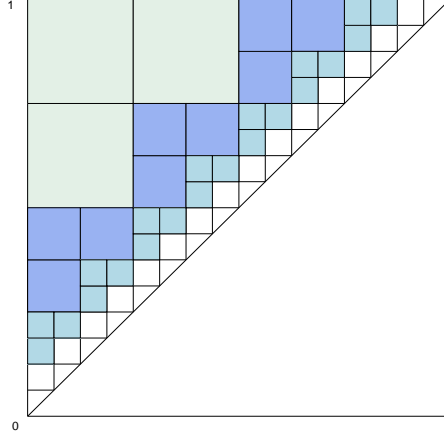
which yields (A.3), as desired. □

Figure 2: Whitney's decomposition in the triangle.

We will use Whitney's decomposition to the triangle $\{(x, y) \in [0, 1]^2 : x < y\}$ (see Figure 2). First, we divide $[0, 1]^2$ into four congruent squares, consider the square with side-length $1/2$ in the triangle region and decompose it into four dyadic squares with side-length $1/4$, then remove the left-upper three ones in the triangle region. Secondly, considering the remaining region, we can find three squares with side-length $1/4$ in the triangle. We decompose each square into four dyadic squares in the same way as in the first step. Repeating the procedure above to the end. So, we have decomposed the triangle region into infinite squares with dyadic border. Let I and J be the dyadic subintervals of $[0, 1]$ in the horizontal and perpendicular axes, respectively. We say that $I \sim J$ if they can consist the horizontal border and perpendicular border of a square described above, respectively. From the decomposition above we see that

(i) $|I| = |J|$ and $\mathrm{dist}(I, \ J) \geqslant |I|$ for $I \sim J$.

(ii) The squares in $\{(x, y) \in [0, 1]^2 : x < y\}$ are pairwise disjoint.

(iii) For any dyadic subinterval J, there are at most two I with $I \sim J$.

Proof of Proposition A.1. First, we show the result of (1). We have

$$T_{re}f(t, x) := \int_{-\infty}^{t} K(t, t')f(t')dt' = \sum_{\{I, J : I \sim J\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f). \qquad (A.17)$$

40

It follows that

$$\|T_{re}f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \left\| \sum_{\{I,J:I\sim J,|I|=2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)}f) \right\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)}. \quad (A.18)$$

For any $p \geqslant 1$, we easily see the following fact:

$$\left\| \sum_{\{I,J:I\sim J,|I|=2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)}f) \right\|_{L_t^p(\mathbb{R})}^p$$
$$\leq 2 \sum_{J_1:|J_1|=2^{-j}} \int_{\mathbb{R}} \chi_{F^{-1}(J_1)} |T(\chi_{F^{-1}(J_1)}f)|^p dt. \quad (A.19)$$

Hence, in view of (A.18) and (A.19) we have

$$\|T_{re}f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \left\| \left(\sum_{\{I:|I|=2^{-j}\}} \|T(\chi_{F^{-1}(I)}f)\|_{L_t^{p_3}(\mathbb{R})}^{p_3} \right)^{1/p_3} \right\|_{L_{x_1}^{p_1} L_{x_2}^{p_2}(\mathbb{R}^2)}. \quad (A.20)$$

If $p \leq q$, by Minkowski's inequality, we have

$$\left\| \left(\sum_j \|a_j(x,y)\|_{L_x^p}^p \right)^{1/p} \right\|_{L_y^q} \leq \left(\sum_j \|a_j(x,y)\|_{L_y^q L_x^p}^p \right)^{1/p}; \quad (A.21)$$

If $p > q$, in view of $(a+b)^{\theta} \leq a^{\theta} + b^{\theta}$ for any $0 \leq \theta \leq 1, a, b > 0$, we have

$$\left\| \left(\sum_j \|a_j(x,y)\|_{L_x^p}^p \right)^{1/p} \right\|_{L_y^q} \leq \left(\sum_j \|a_j(x,y)\|_{L_y^q L_x^p}^q \right)^{1/q}. \quad (A.22)$$

We divide our discussion into the following three cases.

Case 1. $p_1, p_2 \geqslant p_3$. By (A.20), using (A.21) twice, we have

$$\|T_{re}f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \left(\sum_{\{I:|I|=2^{-j}\}} \|T(\chi_{F^{-1}(I)}f)\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)}^{p_3} \right)^{1/p_3}. \quad (A.23)$$

Case 2. $p_1 \leq p_2 \leq p_3$. By (A.20), using (A.22) twice, we have

$$\|T_{re}f\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \left(\sum_{\{I:|I|=2^{-j}\}} \|T(\chi_{F^{-1}(I)}f)\|_{L_{x_1}^{p_1} L_{x_2}^{p_2} L_t^{p_3}(\mathbb{R}^3)}^{p_1} \right)^{1/p_1}. \quad (A.24)$$

Case 3. $p_2 \leq p_1 \leq p_3$. By (A.20) and (A.22), then applying (A.21), we have

$$\|T_{re}f\|_{L_{x_1}^{p_1}L_{x_2}^{p_2}L_t^{p_3}(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty}\left(\sum_{\{I:|I|=2^{-j}\}}\|T(\chi_{F^{-1}(I)}f)\|_{L_{x_1}^{p_1}L_{x_2}^{p_2}L_t^{p_3}(\mathbb{R}^3)}^{p_2}\right)^{1/p_2}. \qquad (A.25)$$

Denote $p_{\min} = \min(p_1, p_2, p_3)$. It follows from (A.23)–(A.25) that

$$\|T_{re}f\|_{L_{x_1}^{p_1}L_{x_2}^{\infty}L_t^{p_2}} \lesssim \sum_{j=1}^{\infty}\left(\sum_{\{I:|I|=2^{-j}\}}|I|^{\frac{p_{\min}q_2}{q_1q_3}\wedge\frac{p_{\min}}{q_3}\wedge\frac{p_{\min}}{q_2}\wedge\frac{p_{\min}}{q_1}}\right)^{\frac{1}{p_{\min}}}$$

$$\lesssim \sum_{j=1}^{\infty}2^{-j((\frac{q_2}{q_1q_3}\wedge\frac{1}{q_3}\wedge\frac{1}{q_2}\wedge\frac{1}{q_1})-\frac{1}{p_{\min}})} < \infty. \qquad (A.26)$$

The proof of (4) is almost the same as that of (1) and we omit the details of the proof.

Next, we prove (2). We have

$$\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}\times\mathbb{R}^2)} \leqslant \sum_{j=1}^{\infty}\left\|\sum_{\{I,J:I\sim J,|I|=2^{-j}\}}\chi_{F^{-1}(J)}T(\chi_{F^{-1}(I)}f)\right\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)}$$

$$\leqslant 2\sum_{j=1}^{\infty}\left\|\sum_{\{I:|I|=2^{-j}\}}\chi_{F^{-1}(J)}\|T(\chi_{F^{-1}(I)}f)\|_{L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^2)}\right\|_{L_t^{p_1}(\mathbb{R})}.$$

Using the same way as in (A.19),

$$\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)} \lesssim \sum_{j=1}^{\infty}\left(\sum_{\{I:|I|=2^{-j}\}}\|\chi_{F^{-1}(I)}f\|_{L_{x_1}^{q_1}L_{x_2}^{q_2}L_t^{q_3}(\mathbb{R}^3)}^{p_1}\right)^{1/p_1},$$

So, we can control $\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)}$ by the right-hand side of (A.26) in the case $p_{\min} = p_1$.

Finally, we prove (3). We define $F_1(t)$ as follows.

$$F_1(t) := \int_{-\infty}^{t}\|f(s, x_1, x_2)\|_{L_{x_1}^{q_2}L_{x_2}^{q_3}}^{q_1}ds. \qquad (A.27)$$

From the definition of $F_1(t)$, it is easy to see that

$$\left\|\chi_{F_1^{-1}(I)}(s)f(s)\right\|_{L_t^{q_1}L_{x_1}^{q_2}L_{x_2}^{q_3}(\mathbb{R}\times\mathbb{R}^2)} = |I|^{1/q_1}. \qquad (A.28)$$

Hence, replacing (A.3) with (A.28), we can use the same way as in the proof of (1) to get the result, as desired. $\qquad \square$

We can generalized this result to n dimensional spaces:

Lemma A.3 *Let T be as in* (A.1). *We have the following results.*

(1) *If $\min(p_1, p_2, p_3) > \max(q_1, q_2, q_3, \ q_1 q_3/q_2)$, then $T : L_{x_1}^{q_1} L_{x_2,\ldots,x_n}^{q_2} L_t^{q_3}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1} L_{x_2,\ldots,x_n}^{p_2} L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*

(2) *If $p_0 > (\vee_{i=1}^3 q_i) \vee (q_1 q_3/q_2)$, then $T : L_{x_1}^{q_1} L_{x_2,\ldots,x_n}^{q_2} L_t^{q_3}(\mathbb{R}^{n+1}) \to L_t^{p_0} L_{x_1}^{p_1} \ldots L_{x_n}^{p_n}(\mathbb{R}^{n+1})$ is a well restriction operator.*

(3) *If $q_0 < \min(p_1, p_2, p_3)$, then $T : L_t^{q_0} L_{x_1}^{q_1} \ldots L_{x_n}^{q_n}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1} L_{x_2,\ldots,x_n}^{p_2} L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*

(4) *If $\min(p_1, p_2, p_3) > \max(q_1, q_2, q_3, \ q_1 q_3/q_2)$, then $T : L_{x_2}^{q_1} L_{x_1,x_3,\ldots,x_n}^{q_2} L_t^{q_3}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1} L_{x_2,\ldots,x_n}^{p_2} L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*

References

[1] J. Bergh and J. Löfström, Interpolation Spaces, Springer–Verlag, 1976.

[2] I. Bejenaru and D. Tataru, Large data local solutions for the derivative NLS equation, arXiv:math.AP/0610092 v1.

[3] H. Chihara, Global existence of small solutions to semilinear Schrödinger equations with guage invariance, Publ. RIMS, **31** (1995), 731–753.

[4] H. Chihara, The initial value problem for cubic semilinear Schrödinger equations with guage invariance, Publ. RIMS, **32** (1996), 445–471.

[5] M. Christ, Illposedness of a Schrödinger equation with derivative regularity, Preprint.

[6] M.Christ, A.Kiselev, Maximal functions associated to filtrations, J. Funct. Anal., **179** (2001), 406-425.

[7] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc., **1** (1988), 413–446.

[8] P. A. Clarkson and J. A. Tuszyriski, Exact solutions oft he multidimensional derivative nonlinear Schrödinger equation for many-body systems near criticality, J. Phys. A: Math. Gen. **23** (1990), 4269-4288.

[9] Wei-Yue Ding, You-DeWang, Schrödinger flow of maps into symplectic manifolds, Science in China Ser. A **41** (1998), 746-755.

[10] J. M. Dixon and J. A. Tuszynski, Coherent structures in strongly interacting many-body systems: II. Classical solutions and quantum fluctuations, J. Phys. A: Math. Gen. **22** (1989), 4895-4920.

[11] H. G. Feichtinger, Modulation spaces on locally compact Abelian group, Technical Report, University of Vienna, 1983. Published in: "Proc. Internat. Conf. on Wavelet and Applications", 99–140. New Delhi Allied Publishers, India, 2003. http://www.unive.ac.at/nuhag-php/bibtex/ open_files/fe03-1_modspa03.pdf.

[12] A. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, II: Global well posedness in dimensions $d \geqslant 3$, Commun. Math. Physics, **271** (2007), 523–559. arXiv:math/0605209v1.

[13] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., **40** (1991), 253–288.

[14] C. E. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrodinger equation, Ann. Inst. Henri Poincaré, Sect C, **10** (1993), 255-288.

[15] C. E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math., **134** (1998), 489–545.

[16] C. E. Kenig, G. Ponce, L. Vega, The Cauchy problem for quasi-linear Schrodinger equations, Invent. Math. **158** (2004), 343–388.

[17] C. E. Kenig, G. Ponce, C. Rolvent, L. Vega, The genreal quasilinear untrahyperbolic Schrodinger equation, Advances in Mathematics **206** (2006), 402–433.

[18] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations, Arch. Rational Mech. Anal., **78** (1982), 73–98.

[19] S. Klainerman, G. Ponce, Global small amplitude solutions to nonlinear evolution equations, Commun. Pure Appl. Math., **36** (1983), 133–141.

[20] L.Molinet, F.Ribaud, Well-posedness results for the generalized Benjamin-Ono equation with small initial data, J.Math.Pures Appl., **83** (2004), 277-311.

[21] T. Ozawa and J. Zhai, Global existence of small classical solutions to nonlinear Schrödinger equations, Ann. I. H. Poincaré, AN, **25** (2008), 303–311.

[22] J. Shatah, Global existence of small classical solutions to nonlinear evolution equations, J. Differential Equations, **46** (1982), 409–423.

[23] P. Sjölin, Regularity of solutions to the Schrödinger equations, Duke Math. J., **55** (1987), 699–715.

[24] H. F. Smith, C. D. Sogge, Global Strichartz estimates for nontrapping perturbations of Laplacian Comm.Partial Differential Equations **25** (2000), 2171-2183.

[25] M. Sugimoto amd N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, J. Funct. Anal., **248** (2007), 79–106.

[26] J. A. Tuszynski and J. M. Dixon, Coherent structures in strongly interacting many-body systems: I. Derivation of dynamics, J. Phys. A: Math. Gen. **22** (1989), 4877-4894.

[27] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I. J. Funct. Anal., **207** (2004), 399–429.

[28] H. Triebel, Theory of Function Spaces, Birkhäuser–Verlag, 1983.

[29] B. X. Wang, L. F. Zhao and B. L. Guo, Isometric decomposition operators, function spaces $E_{p,q}^{\lambda}$ and applications to nonlinear evolution equations, J. Funct. Anal., **233** (2006), 1–39.

[30] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations, **231** (2007), 36–73.

[31] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Differential Equations, **239** (2007), 213–250.

[32] B. X. Wang and Y. Z. Wang, Global well posedness and scattering for the elliptic and non-elliptic derivative nonlinear Schrödinger equations with small data, Preprint, arXiv:0803.2634.

[33] L. Vega, The Schrödinger equation: pointwise convergence to the initial data, Proc. Amer. Math. Soc., **102** (1988), 874–878.